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Decision Analysis Incorporating Preferences of Groups

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DECISION ANALYSIS INCORPORATING PREFERENCES OF GROUPS

by

Craig W. Kirkwood

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The theoretical development divides into three parts. First, methods are developed for combining the preferences of various individuals into a single description of the preferences of the entire group. Second, new methods are developed for assessing the preferences of the different individuals. Finally, a Bayesian approach is given for incorporating into the analysis the decision makers' uncertainty about the preferences of the individuals of interest. (U)

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II

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by

CRAIG W. KIRKWOOD

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Adapted from a thesis, supervised by Professor Ralph L. Keeney, presented to the Department of Electrical Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy, June, 1972.

FOREWORD

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John D. C. Little
Director

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Craig W. Kirkwood

ABSTRACT

Individuals or small groups often make decisions which affect the interests of other people. The decision makers may wish to incorporate the preferences of these people into their analysis of alternative courses of action. A normative methodology for doing this, using results from decision theory, is developed in this report.

The theoretical development divides into three parts. First, methods are developed for combining the preferences of various individuals into a single description of the preferences of the entire group. Second, new methods are developed for assessing the preferences of the different individuals. Finally, a Bayesian approach is given for incorporating into the analysis the decision makers' uncertainty about the preferences of the individuals of interest.

The methodology is applied to three "real-world" situations. One of these shows its use in providing direct citizen participation in local government decision making. The second application demonstrates how computer time-share system managers could use the methods to incorporate the preferences of the system users into their planning process. Finally, it is shown how government planners could use the methodology to incorporate the preferences of the affected people into the planning of new housing to replace that destroyed by highway construction.

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INTRODUCTION

In modern government and industrial operations individuals or small groups often make decisions that affect the lives of many different individuals and groups. Thus, for example, a decision by the Boston-area Metropolitan District Commission on how to meet future demand for sewage treatment facilities will affect the quality of life of residents of the city. The waste disposal problem for various groups within the city will be met to a greater or lesser extent depending on what plan is adopted. Also, of course, the cost to various groups will differ depending on what plan is adopted.

To take another example a decision as to what program to institute in a community to fight heroin use will affect many different groups within the community. Addicts will receive different treatment depending on what program is selected; the citizenry will be protected from addict-committed crimes to differing degrees depending on the program selected; and drug pushers will be affected differently.

In the private sector, many companies are becoming more concerned with the ways in which their decisions affect different community groups and the government's opinion. Thus, for example, power companies are becoming aware of the ways in which their methods of electric power generation affect different community residents.

Clearly, the detailed structure of these problems differs greatly. However, they all share the common feature that the results of the decision

will impact on a number of different people or groups in different ways. Furthermore, a decision maker in such situations often wishes to take the preferences of the affected people into account while making his decision. In this thesis methods are developed for formally doing this.

The approach taken is to identify a number of problems that a decision maker faces when he wishes to incorporate the preferences of other people into his analysis, and then to develop a methodology for tackling these problems. After doing this the methodology is applied to several problems in order to demonstrate its strengths and weaknesses.

1.1 Difficulties Associated with Incorporating Preferences of Others into an Analysis

There are two types of difficulties that arise when an attempt is made to incorporate other people's preferences into an analysis. First, there are fundamental theoretical difficulties that limit the manner in which this may be done. Second, there are a number of practical difficulties that make it hard in any realistic situation. These two types of difficulties will now be considered.

1.1.1 Fundamental Theoretical Difficulties

One basic difficulty is determining how the preferences of different individuals may be compared in a meaningful way. This problem is well illustrated by the following simple example.[1]

Suppose a host wishes to serve his two guests coffee, tea or hot chocolate. Since he has only one pot in which to make the beverage he must serve the same thing to both guests. To help him decide which to serve he asks the guests to rank the drinks in order by preference. Suppose guest one

responds that he likes coffee best, then tea and finally hot chocolate. Guest two, on the other hand, ranks them hot chocolate, tea and coffee.

Based on these lists the host might reasonably feel he should serve tea. Both guests like this second best, while they split in their preferences for the other two drinks, one preferring coffee and liking hot chocolate least while the other's preferences are just the opposite.

However, further thought leads to the conclusion that this may not be the best course of action to take. The ranking lists do not tell anything about the relative <u>intensity</u> of the two guests' preferences. For example, guest one may be fairly satisfied with any of the drinks while guest two may dislike tea and coffee very much.

In order to obtain this intensity information the host might ask each guest to rank the drinks on a 1 to 10 scale where 1 means "dislike greatly" and 10 means "like very much." Suppose guest one scores the drinks as coffee = 6, tea = 4 and, hot chocolate = 3, while guest two lists hot chocolate = 8, tea = 7, and coffee = 6.

Looking at this list the host might feel he should serve coffee. After all, both guests like this fairly well (as measured by this method) and any other drink is liked considerably less by guest one.

But, further consideration leads to the realization that this may not be the best drink to serve. It is not clear what the various numbers on the 1 to 10 scale mean to each of the guests. Thus, for example, to guest one a rank of 1 might be assigned to a drink that he would just barely consider drinking while

guest two might reserve ranks 1 to 3 for drinks that he would not ever consider drinking. If this were true, guest one would tend to assign higher ranks to drinks that he had the same innate preference for as guest two.

By careful questioning it would be possible to decrease the ambiguity in the meaning of the scale values. However, this can never be resolved completely. At some point the host will have to use his own judgment to decide what scale values represent the same level of preference for the two different individuals.

This difficulty of interpersonally comparing preferences seems to be fundamental and inescapable. There is no objectively correct way to compare the preferences of different people. Or, as Bergson said in his classic paper:[2] "No extension of the methods of measuring utilities will dispense with the necessity for the introduction of value propositions to give these utilities a common dimension."

Although this problem is inescapable, any methodology that is to be useful to practical decision makers should reduce the number of such value judgments that must be made. It should also bring these judgments out into the open and explicitly show the affects of possible changes in them. This will provide a means for persons who disagree about these judgments to investigate the affects of their disagreements on the analysis.

Even if the problem of interpersonal comparison of preferences were satisfactorily resolved, the question of how much the preferences of different individuals or groups should be counted by the decision maker still remains.

If it is agreed in the coffee-tea-hot chocolate example above that the 1 to 10 scale for each of the guests represent the same innate preferences, the host might still take their preferences into account to different degrees. Thus, for example, if both guests were equally good friends he might try to reach a compromise on the drink served. However, if one guest were the host's employer, he might be guided totally by that guest's preferences.

To take a more serious example, suppose a school board is trying to choose a plan for a new school. The board might wish to take into account the preferences of various community members differently depending on whether they have children in school or not.

This question of how much "weight" to give to the views of different individuals or groups does not have an objective solution. It will clearly depend on the decision being made and the person making it. However, any practical methodology for incorporating the preferences of others into an analysis should allow for an open display of the weight being given to different people's preferences so that the affect of changes in the weights can be seen.

Both the problem of interpersonal comparison of preferences and of relative weight to be given to the views of different individuals or groups are discussed in chapter IV of this thesis. The methods developed there will aid decision makers in tackling these problems.

1.1.2 <u>Practical Difficulties</u>

Even if the theoretical difficulties discussed in the last section are overcome, there are two important practical problems that must be tackled

before a decision maker can incorporate the preferences of others into his analysis.

First, if the number of people whose views are of interest to him is large there may not be time or resources to assess all their preferences. Second, even if their preferences are obtained, the assessed values may not represent the views of the individuals accurately. This may be due to a deliberate attempt to conceal true preferences or because the individuals haven't thought carefully enough about what their preferences are. This problem will be particularly acute if the views of a large number of people are to be obtained. In that case it becomes difficult to spend the time with each person needed to properly determine his views.[3]

In chapter V a method is discussed that helps to cut down the time needed to assess a person's preferences and, at the same time, makes it easier to check whether the views obtained represent the "true" preferences of the individual.

In chapter VI ways of dealing with uncertainty about the preferences of the individuals or groups of interest are presented. Uncertainty due to possible sampling error as well as that resulting from inaccurate statement of preferences by the individuals asked is considered.

1.2 Basic Approach of the Thesis

From the last section it is apparent that two key features involved in incorporating the preferences of others into a decision maker's analysis are that there is uncertainty and that the preferences of people are of interest.

The decision theory of von Neumann and Morgenstern[4] was established expressly to deal with problems involving uncertainty and preferences. Therefore a number of results from this theory will be useful in this thesis research.

In particular, some recent results from multiattribute utility theory will be very valuable. (For those unfamiliar with decision theory, a summary is included in chapter III.)

In multiattribute decision theory, the preferences of a person are summarized by a utility function U. This depends on attributes $x_1, x_2, \ldots, x_m$ that describe possible states of the world; that is $U = U(x_1, x_2, \ldots, x_m)$. Since this thesis studies situations where the decision maker's preferences depend on the views of other individuals or groups, it follows that

$$U = U(x_1, x_2, \ldots, x_m; u_1, u_2, \ldots, u_n) \qquad (1.2.1)$$

where $u_1, u_2, \ldots, u_n$ are the utilities of the individuals or groups whose views are important to him. That is, the decision maker's utility function depends on the utility functions of other individuals or groups as well as directly on the attributes $x_1, x_2, \ldots, x_m$.

Using this basic approach, the thesis presents methods for assessing U. In addition, procedures for determining the u_i's in practical situations are presented. Also, the problem of dealing with uncertainty in the u_i's is discussed.

Throughout the thesis, the emphasis is on deriving results that will be useful to real-world decision makers.

1.3 Outline of the Thesis

Chapter II presents past research results concerned with the use of decision theory to incorporate the preferences of the members of a group into a decision making process. The results that will be discussed deal mainly with the theoretical difficulties of interpersonal comparison of preferences and weighting of different people's views that were noted in section 1.1.1. Much past research has been done on resolving the practical difficulties of preference assessment and incomplete information discussed in section 1.1.2. However, none of the past approaches have used decision theory so they are not very helpful as background for the current study.

Chapter III reviews basic ideas of decision theory with particular emphasis on multiattribute utility theory. It also presents a summary of the more important results in this thesis.

Chapter IV considers in detail how the decision maker's utility function $U(x_1, x_2, \ldots, x_m; u_1, u_2, \ldots, u_n)$ can be assessed.

Chapters V and VI develop methods to tackle the practical problems discussed in section 1.1.2. Chapter V discusses the assessment of preferences for people whose views are of interest to the decision maker. The emphasis here is on procedures that yield reasonable approximations to the preferences while still being operationally feasible. Chapter VI considers ways of handling uncertainty in the preferences of the people of interest. Both the case where the uncertainty is due to inability to assess the preferences of everyone and

the case where there is possible error in the assessed preferences are considered.

Chapter VII considers three applications of the results derived in the thesis. One of these considers methods that a person interested in the preferences of a discussion group could use to obtain these preferences. The second example considers a way of assembling the preferences of various users of a time-share computer system. The third example discusses the assessment of various proposed sites for building new housing for families that will be displaced by highway construction. It considers how the preferences of the displaced families may be taken into account.

Finally, Chapter VIII discusses further research that might be carried out building on the work in this thesis.

The reader interested in an overview of the thesis research may read chapter III and one or more of the examples in chapter VII. The reader interested in the technical details of the thesis results should also read chapters IV, V and VI.

Chapter I Footnotes

1. See Luce and Raiffa[22], ch. 14, for a longer discussion of this difficulty.

2. See Bergson[4], p. 327.

3. Grochow[9] has noted how time consuming this is.

4. For a discussion of this see von Neumann and Morgenstern[36], North[25], Howard[13], Raiffa[28], or Pratt, Raiffa and Schlaifer[27].

Chapter II

BACKGROUND

Much of the previous work on the use of decision theoretic methods in the assessment of the preferences of groups has been done by welfare economists. Henderson and Quandt[1] say that

> the objective of welfare economics is the evaluation of the social desirability of alternative economic states. An economic state is a particular arrangement of economic activities and of the resources of the economy.

Many welfare economists feel that this evaluation can only be done reasonably if the preferences of the members of the society are used in the analysis. Hence they have been concerned with ways of obtaining these preferences and combining them to give a measure of the overall preferences of the society for different alternatives.

In this chapter a number of results obtained by previous researchers will be discussed. Before doing this some useful notation is presented.

Let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of possible alternative states under consideration in a particular decision analysis. Further, let U be a utility function representing the preferences of the group of people as a whole and let $u_1, u_2, \ldots, u_n$ be the utility functions of the members of the group.[2] Thus $U(a_i)$ is the utility of alternative a_i to the group as a whole while $u_j(a_i)$ is the utility of that alternative to the $j^{\underline{th}}$ individual in the group.

Since welfare economists usually call U a <u>social welfare function</u>, that term will be used here sometimes.

2.1 Ranking Schemes to Construct Social Welfare Functions

One class of welfare functions that has been widely studied is based on ranking lists of the preferences of the people in the group. Each person gives a list of the possible alternatives ranked according to his preference for them and these lists are combined to give a ranking list for the group as a whole.

This scheme does not take into account the intensity of the individuals relative preferences for different alternatives. However, it has the advantage of being easily explainable to the people whose preferences are desired. Furthermore, it is a generalization of the standard voting procedure used to select officeholders in many groups. Thus it is a natural procedure to use to obtain preferences from a group.

A number of different rules have been proposed for combining the ranking lists of the group members to obtain a ranking list for the total group. The simplest is probably majority rule. When this is used the alternatives are compared pairwise. For each pair the one which is preferred by more people is said to be the more preferred of the two by the group. (In the case of a tie both alternatives are said to be equally preferred.) From these pairwise preference orderings an attempt is made to construct a ranking list for the group.

For example, suppose there are three individuals I, II and III and three possible alternatives a_1, a_2 and a_3. Suppose the ranking lists for the three individuals are

$$\text{I:}\ a_1 \succ a_2 \succ a_3,$$

$$\text{II:}\ a_1 \succ a_2 \sim a_3, \qquad (2.1.1)$$

and

$$\text{III:}\ a_3 \succ a_1 \succ a_2,$$

where "$\succ$" is read "is preferred to" and "$\sim$" is read "is indifferent to." Then by majority rule the group prefers a_1 to a_2, a_1 to a_3, and is indifferent between a_2 and a_3. Therefore, a ranking list for the group as a whole is

$$a_1 \succ a_2 \sim a_3. \qquad (2.1.2)$$

Unfortunately it isn't always possible to combine the pairwise preference rankings to obtain a ranking list for the group. For example, Condorcet pointed out in the eighteenth century[3] that if there are three individuals with preference rankings

$$\text{I:}\ a_1 \succ a_2 \succ a_3,$$

$$\text{II:}\ a_2 \succ a_3 \succ a_1,$$

and

$$\text{III:}\ a_3 \succ a_1 \succ a_2,$$

then majority rule does not give a ranking list for the group. Majority rule says that a_1 is preferred to a_2 by the group, a_2 is preferred to a_3, and a_3 is preferred to a_1. This set of pairwise rankings is intransitive and cannot be organized into a ranking list.

The work on ranking schemes to construct social welfare functions culminated in Arrow's 1951 monograph. This is discussed in the next section.

2.1.1 Arrow's General Possibility Theorem

Arrow proved that if a few fairly reasonable constraints are imposed on the way in which individual ranking lists are combined into a group ranking list, then there is <u>no</u> procedure that can be used to combine the individual lists that can be guaranteed to yield a ranking list for the group.[4]

The constraints that Arrow imposed were:[5]

1. a) There are at least three possible alternatives in the set A of possible alternatives.

   b) The social ranking list is defined for all possible individual ranking lists.

   c) There are at least two individuals.

2. (Positive association of social and individual values.) If the social ranking list asserts that a_i is preferred to a_j for a given profile of individual ranking lists, it should assert the same if the profile is modified as follows:

   a) The individual paired comparisons between alternatives other than a_i and a_j are not changed

   and

   b) Each individual paired comparison between a_i and any other alternative either remains unchanged or is modified in a_i's favor.

3. (Independence of irrelevant alternatives.) Suppose A_1 is any subset of states in A. If a profile of ranking lists is modified in such a manner that each individual's paired comparisons among the states of A_1 are left unchanged, the social rankings resulting from the original and modified profiles of individual rankings shall be identical for all the alternatives in A_1.

4. (Citizen's sovereignty.) For each pair of states a_i and a_j, there is some profile of individual rankings such that the group prefers a_i to a_j.

5. (Non-dictatorship.) There is no individual such that whenever he prefers a_i to a_j (for any a_i and a_j) the group does likewise, regardless of the preferences of other individuals.

Although all of these five constraints seem reasonable some objections have been raised to 3 (independence of irrelevant alternatives).[6] Basically the objections say that the "irrelevant" alternatives are not really irrelevant because they allow the group members to show the strength of their preferences for the different relevant alternatives.

Objections have also been raised to constraint 1b. It seems somewhat stringent to require that the ranking procedure work for every possible set of individual ranking lists. Arrow considered one restriction on the individual ranking lists that does allow a group list to be constructed. This is considered in the next section.

2.1.2 The Single-peakedness Condition

Arrow has shown[7] that if the preferences of the individuals in a group obey the "single-peakedness" condition and if there is an odd number of people in the group then majority rule is a method of combining individual ranking lists into a group ranking list which meets the five constraints in the last section except 1b.

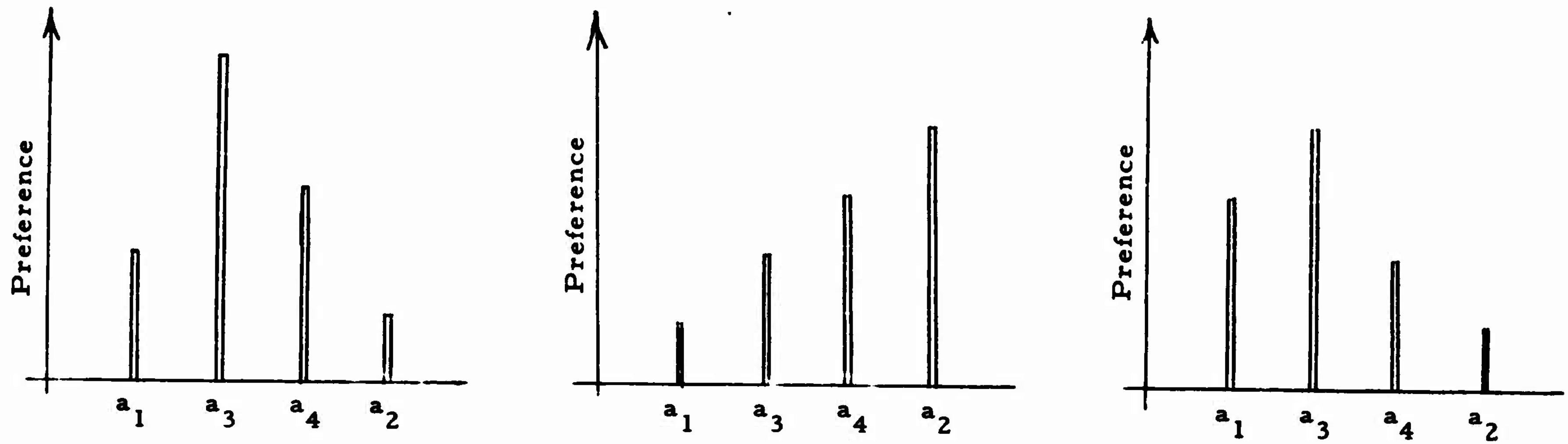
The single-peakedness condition says that there is a scale along which the possible alternatives $a_1, a_2, \ldots, a_n$ may be arranged (not necessarily in numerical order) such that a graph of the relative preferences for the various alternatives for <u>each</u> individual in the group has a single peak. (Note that the same arrangement of states along the scale must be used for <u>every</u> individual in the group although the peak may be in different places for different individuals.

For example, suppose there are four alternatives and three individuals. Then the preference profiles in figure 2.1a obey the single-peakedness condition. However, those in 2.1b do not. Individual 3's preference profile has two peaks. Any rearrangement of the states along the horizontal scale to eliminate the second peak will create a second peak in the preference profile of one of the other individuals.

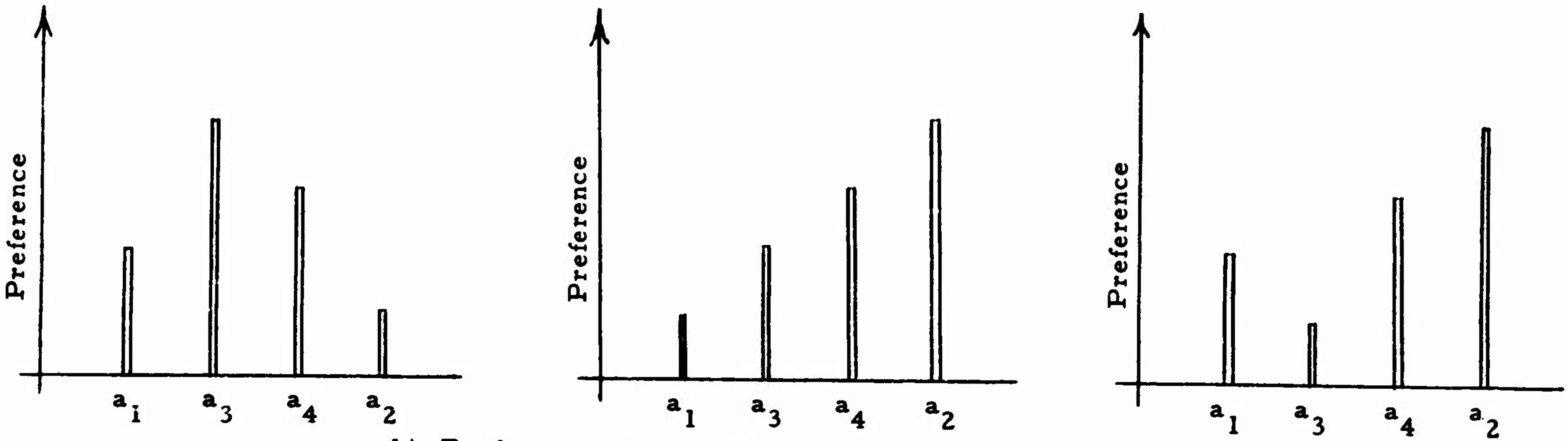
The single-peakedness condition places great restrictions on the allowable preference patterns of the group members. Although it would not be obeyed by the preferences of most groups Arrow does present some cases where it might hold.

2.2 Fleming's Theorem

Fleming has established conditions under which an ordinal social welfare function may be written as the sum of the ordinal utility functions for the group members.[8] To be specific, if the five conditions given below are satisfied then there exist real-valued functions $f_1, f_2, \ldots, f_n$ on $A = \{a_1, a_2, \ldots, a_m\}$ such that



a) Preferences Obey Single-peakedness Conditions



b) Preferences do not Obey Single-peakedness Condition

Figure 2.1. Example of Single-peakedness Condition

1. Individual i prefers a_j to a_k if and only if $f_i(a_j) > f_i(a_k)$, and

2. The group as a whole prefers a_j to a_k if and only if $f(a_j) > f(a_k)$ where

$$f(a_j) = \sum_{i=1}^{n} f_i(a_j).$$

Note that this result does not say that the sum of any individual ordinal utility functions may be used as a social welfare function. Rather, it says that there are some ordinal utility functions whose sum may be used as a social welfare function.

The conditions necessary for the result to hold are:[9]

1. (Asymmetry of group preferences.) If the group as a whole prefers a_k to a_j then it does not prefer a_j to a_k.

2. (Transitivity of group preferences.) If the group prefers a_i to a_j and a_j to a_k then it prefers a_i to a_k.

3. (Transitivity of group indifference.) If the group is indifferent between a_i and a_j and between a_j and a_k then it is indifferent between a_i and a_k.

4. (Positive relation of group preferences to individual preferences.) If one individual prefers a_i to a_j, and none of the other individuals prefer a_j to a_i then a_i is preferred to a_j by the group.

5. (Independent evaluation of the utility distribution between each pair of individuals.)

a) There are at least three individuals in the group.

b) Suppose all the members of the group except two are indifferent among a set of possible alternatives. Then the group preferences among the alternatives depend only on the two individuals who are not indifferent.

Fleming proves the result by constructing the functions $f_1, f_2, \ldots, f_n$. However, it would be difficult to carry out this construction in an actual situation because the person assessing the f_i's must interpersonally compare the utilities of different individuals for many different alternatives. This is difficult to do and it seems unlikely that he could do it and have very much confidence in his results.

Furthermore, because of this interpersonal comparison, the assessed group utility function would be dependent in a complex manner on the value structure of the person doing the assessment. It was noted in chapter I that a group utility function always depends on the value structure of the person constructing it. However, in Fleming's result this dependence is particularly complicated. Therefore it would be difficult to see exactly what the consequences of the assessor's value judgments were.

For these reasons Fleming's work is not much help in actually constructing a group utility function. However, it does provide insight into the problem of amalgamating individual ordinal utility functions to obtain a group ordinal utility function.

Recently Fishburn has establish other conditions that lead to an additive ordinal group utility function.[11] These are relatively mathematical and not as intuitive as Fleming's conditions.

2.3 Goodman-Markowitz Theorem

The work in the last two sections has not taken into account the relative intensity of each group member's preferences for different alternatives. Goodman and Markowitz did this to a certain extent by introducing the idea of "levels of discretion."[12]

They assumed that each person's utility function u_i, $i = 1,2,\ldots,n$, can take on only a finite number L_i, $i = 1,2,\ldots,n$, of different values (or levels of discretion). That is, the $i^{\underline{th}}$ individual could preference rank at most L_i different alternatives before being indifferent between some of them. (Notice that the number of levels of discretion does not have to be the same for different individuals.)

If this idea is accepted and if u_i is assumed to take on only the values $1,2,\ldots,L_i$, then Goodman and Markowitz have shown that the group utility function must be

$$U(a_j) = \sum_{i=1}^{n} u_i(a_j) \tag{2.3.1}$$

if three conditions are imposed. These conditions are:

1. (Pareto-optimality.) If no individual prefers a_j to a_i and if at least one individual prefers a_i to a_j then the group prefers a_i to a_j.

2. (Symmetry.) The group ordering of alternatives is unchanged if the utilities of any two individuals for all the alternatives are interchanged.

3. Suppose individual i has L_i levels of discretion. Then the social ordering between two alternatives a_k and a_ℓ is unchanged if $u_i(a_k)$ and $u_i(a_\ell)$ are replaced with $u_i(a_k) + c$ and $u_i(a_\ell) + c$ where

$$1 \leq u_i(a_j) + c \leq \max_i [L_i]$$

for all j.

The most questionable of these conditions is probably 2. This says, in essence, that a level of discretion represents the same preference shift for any person. This seems unreasonable in many cases. Suppose, for example, that one individual considers all alternatives to be either "acceptable" or unacceptable," while another has many gradations of preference. It is not clear that the levels of discretion of these two individuals should be counted equally.

Goodman and Markowitz recognized this objection and noted that if condition 2 is removed than the social welfare function must be of the form

$$U(a_j) = \sum_{i=1} w_i u_i(a_j) \tag{2.3.2}$$

where the w_i's are constants that are positive but otherwise arbitrary.

From an operational point of view Goodman and Markowitz's result is difficult to use because there seems to be no way to determine the levels of

discretion of an individual. These might be approximated by asking him to rank a very large number of different alternatives. However, this is a very cumbersome procedure and it would never be certain that all the levels had been found.

Thus, like Fleming's theorem, this result is not very operationally useful. However, it does provide more insight into the difficulties of constructing utility functions to represent the preferences of a group.

2.4 Cardinal Utility and Social Welfare Functions

It was noted in section 1.1.2 that decision makers wishing to incorporate preferences of others into their analysis often must cope with uncertainty. In such situations cardinal utility functions are of more use than ordinal ones like those that have been studied in the last three sections.[13]

Two interesting results involving cardinal utility functions have been derived by Nash and Harsanyi. These are discussed in the next two sections.

2.4.1 Nash's Theorem

Nash was originally concerned with the two-person bargaining situation.[14] He set down conditions that an arbitration scheme to settle the bargaining problem should obey and derived a solution involving the cardinal utilities of the two individuals.

Luce and Raiffa noted the similarity of this problem to the social welfare problem.[15] They generalized Nash's work to groups larger than two people and pointed out its interpretation in the context of social welfare. They assumed that the cardinal utility functions u_i, $i = 1, 2, \ldots, n$ of the group

members were known over the possible alternatives $a_o, a_1, \ldots, a_m$, where a_o represents the status quo or "do nothing" decision.

Then, if the conditions given below are imposed and if there is at least one a_j such that $u_i(a_j) > u_i(a_o)$ for all i, the alternative a* that should be chosen by the group is the one that maximizes

$$\prod_{i=1}^{n} [u_i(a^*) - u_i(a_o)] \tag{2.4.1}$$

subject to the constraint that $u_i(a^*) > u_i(a_o)$ for all i. (Notice that a* may be a probabilistic mixture of various a_i's if this is allowed.)

The conditions that lead to Nash's solution are:

1. The alternative preferred by the group shall not depend on the utility scales (origins and units of measurement) of the u_i's.

2. (Pareto-optimality.) If a_k is the alternative preferred by the group there shall not be another alternative a_ℓ such that $u_i(a_\ell) \geq u_i(a_k)$ for all i.

3. (Independence of irrelevant alternatives.) Adding new alternatives with a_o kept fixed, shall not change an old alternative from a non-preferred to the preferred alternative for the group.

4. (Symmetry.) Suppose by changing the scale and origin of the individual utility functions it is possible to obtain a description of the decision problem where

   a) $u_i(a_o) = u_j(a_o)$ for all i,j

   and

b) There exists an a_j for any a_i such that the vector

$$[u_i(a_j), u_2(a_j), \ldots, u_n(a_j)]$$

$$= P[u_i(a_i), u_2(a_i), \ldots, u_n(a_i)]$$

where

$$P[u_i(a_i), u_2(a_i), \ldots, u_n(a_i)]$$

is any vector formed by permuting the components of

$$[u_1(a_i), u_2(a_i), \ldots, u_n(a_i)].$$

Then the socially preferred alternative a^* is the one such that

$$u_i(a^*) = u_j(a^*)$$

for all i and j. (Recall, as was mentioned above, that a^* may be a probabilistic mixture of the a_i's.)

It seems from condition 1 that Nash's solution avoids any interpersonal comparison of preferences. However, because of the symmetry condition 4 this is not true. This condition says that if certain symmetry conditions are placed on the problem the solution that gives everyone the same utility should be picked. This is certainly an interpersonal utility comparison.

A number of researchers have raised objections to Nash's solution.[16] Most of these have consisted of examples where it is contended that the Nash solution is not "fair." Enough of these have been found to cast doubt on the usefulness of the Nash solution in practical group decision problems.

2.4.2 Harsanyi's Theorem

Harsanyi discussed some conditions under which a group cardinal utility function may be constructed from the cardinal utility functions of the group members.[17] In particular, he showed that if certain conditions are met then the group utility for any alternative a is given by

$$U(a) = \sum_{i=1}^{n} w_i u_i(a) \tag{2.4.2}$$

where the w_i's are positive constants.

The conditions which lead to (2.4.2) are:

1. The group utility function U(a) obeys the von Neumann-Morgenstern axioms of cardinal utility.[18]
2. The individual utility functions $u_1(a), u_2(a), \ldots, u_n(a)$ also obey these axioms.
3. If two situations are indifferent from the standpoint of each individual, they are also indifferent for the group as a whole.

These conditions seem to be very weak to lead to such a strong result. However, condition 3 is actually fairly strong. Suppose, for example, there are two people in the group and four alternatives a_1, a_2, a_3 and a_4 such that

$$0 = u_1(a_1) = u_2(a_1) = u_1(a_4) = u_2(a_3)$$

and

$$1 = u_1(a_2) = u_2(a_2) = u_1(a_3) = u_2(a_4).$$

Now consider the lotteries

$$L_1 : \langle a_1 ; a_2 \rangle$$

and

$$L_2 : \langle a_3 ; a_4 \rangle$$

where there is a 50:50 chance of either result in each lottery. By Harsanyi's condition 3 it must be true that $U(L_1) = U(L_2)$ since each individual is indifferent between the two lotteries.

However, in many cases it would be reasonable for the group utilities of the two lotteries to be different. In L_1 the members of the group will both end up with equal utilities regardless of which outcome occurs while in L_2 they will both end up with differing utilities regardless of which outcome occurs. It is not clear that either of these situations is always socially desirable, however, it seems that in many cases one or the other would be more desirable. In those cases Harsanyi's condition 3 is violated.

Even if condition 3 is accepted so that equation (2.4.2) holds, the weighting constants $w_1, w_2, \ldots, w_n$ must be assessed. Van den Bogaard and Versluis,[19] and Theil[20] have considered this problem. The interested reader may consult their papers. This problem will also be considered in chapter IV of this thesis.

This concludes the review of past research related to the work in this thesis. The most directly related work is that of Harsanyi. In chapter IV some generalizations of his work will be presented. There some results concerning situations where condition 3 does not hold will be presented.

Chapter II Footnotes

1. See Henderson and Quandt[12], p. 201.

2. The reader unfamiliar with utility functions should read sections 3.1 and 3.2 before continuing with chapter II.

3. Condorcet's work is discussed in Guilbaud[10].

4. See Arrow[3]. A flaw in Arrow's original formulation was pointed out by Blau[5].

5. Luce and Raiffa's[22] formulation of the constraints is used here.

6. See Luce and Raiffa[22], pp. 335-37 for a detailed discussion of these objections.

7. See Arrow[3], pp. 75-80.

8. See Fleming[7] for a proof of this result.

9. The formulation given here follows Harsanyi[11].

10. This is a paraphrase of the actual condition 5b in Fleming's theorem. However, it conveys the essential meaning of that condition.

11. See Fishburn[6].

12. See Goodman and Markowitz[8].

13. Raiffa[28] gives a clear introduction to cardinal utility functions.

14. See Nash[24].

15. See Luce and Raiffa[22], pp. 349-50.

16. See Luce and Raiffa[22], pp. 128-34.

17. See Harsanyi[11].

18. For a discussion of these see von Neumann and Morgenstern[36], pp. 641ff or Pratt, Raiffa and Schlaiffer[27], chs. 2-3.

19. See van den Bogaard and Versluis[35].

20. See Theil[34].

Chapter III

BASIC IDEAS AND SUMMARY OF RESULTS

Because this thesis uses many ideas and results from decision theory a brief summary of this theory is given in the next two sections. This is followed by a detailed discussion of the research approach taken and a summary of the major results obtained.

3.1 Formal Decision Theory

Decision theory[1] assumes that a decision maker can identify the set $A = \{a_1, a_2, \ldots, a_n\}$ of possible actions open to him and the set $O = \{o_1, o_2, \ldots o_m\}$ of possible outcomes from these actions. After identifying A and O the decision maker, if he wishes his reasoning to obey certain "reasonable" conditions,[2] should assign two functions $p_{o|a}(o_i|a_j)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $u(o_i)$, $i = 1, 2, \ldots, m$. The function $p_o(o_i|a_j)$ encodes his feelings about the relative likelihood of the various outcomes occurring given that he carries out a particular action. It is usually called a subjective probability function.

The function $u(o_i)$ encodes the decision maker's relative preferences for the different possible outcomes. It is usually called a (cardinal) utility function.

Decision theory proves that if the decision maker accepts the conditions on his reasoning than he should calculate for each a_j the expected utility

$$E[u(a_j)] = \sum_{i=1}^{m} u(o_i) p_{o|a}(c_i|a_j) \qquad (3.1.1)$$

and select the one with the highest expected utility.

This theory is appealing on theoretical grounds as a normative guide for decision making. However, there are difficulties in using it. Often each possible action and outcome consists of many subparts. The connections among these may be unclear to the decision maker. Furthermore, the various possible outcomes may differ from each other in a number of diverse aspects. This makes it hard for the decision maker to specify with assurance a utility function giving his relative preferences for different outcomes.

Because of this difficulty, decision theory has not been applied extensively. However, in the last few years progress has been made toward developing an applied theory of decision making based on formal decision theory. This applications-oriented field is called decision analysis.

One approach to decision analysis[3] involves the use of multiattribute descriptions of the possible outcomes of a decision making process. Since this approach will be useful for the work in this thesis, it is discussed in some detail in the next section.

3.2 Multiattribute Decision Analysis

Multiattribute decision analysis adapts decesion theory so that it conforms more closely to the manner in which practical decision makers think about their decision problems.

In many decision problems a need is initially perceived in very general terms. For example, a need to relieve overcrowding at a municipal airport might be perceived. Then various broad classes of solutions are proposed. In the airport case, one might expand the present airport, build a new one,

or try to reduce air travel into the airport--perhaps by improving regional ground transportation. Often the different types of solutions will meet the need to differing degrees and, in addition, will have side effects. A new airport might, for example, provide new jobs, increase environmental pollution, and lower the value of the land around the airport.

Multiattribute decision analysis allows the decision maker to formalize the process outlined in the last paragraph. He defines attributes which describe the aspects of the situation that are important for his decision purposes, and then assesses his utility function for various amounts of these attributes. This helps to identify the aspects of the decision problem that are most crucial to him, and thus serves as a useful aid for devising courses of action that will solve the problem.

As the decision maker finds these courses of action he will often discover that they have side effects that were not described by the original attribute set. The attribute set can then be augmented to account for these. The utility of various values of the new attributes can be assessed and the possible courses of action refined into more definite operational plans.

This iterative procedure may be continued through several cycles until a particular plan is decided on.

Of course, while this analysis of the decision maker's preferences is being carried out it is necessary to account for uncertainty regarding the outcome that will result from any course of action. Therefore, probabilistic models must be built, and improved as the analysis proceeds, to describe the

uncertainties in the possible results of the different actions.[4] The construction of such models is a familiar operations research activity.

Although multiattribute decision analysis is easier to apply than formal decision theory there are still problems with using it in realistic situations. In particular, it is difficult for a decision maker to assess a utility function over the attributes of interest. This may not be as hard as assessing utilities directly for outcomes, but it is still difficult.

Recent work by Keeney and Raiffa[5] has provided theoretical tools to help in this assessment. Using these it is usually possible to at least approximate a decision maker's utility function and investigate whether the solution is sensitive to changes in the approximation.

Another difficulty is that there is often not a single decision maker. Thus it is not clear whose utility function or probability functions should be used in the analysis. One way to proceed is to use the different preference and uncertainty judgments of the various people involved and see how they change the results of the analysis. In fact, a decision analysis model provides a good way for people to determine the consequences of their differing judgments of uncertainty and preferences.

The situation of interest in this thesis, where a decision maker wishes to incorporate the preferences of others into his analysis, involves preference judgments and uncertainty. Thus it can usefully be studied using decision analysis.

3.3 Decision Analysis Incorporating Preferences of Groups

In section 1.2 a brief discussion of the basic approach of this thesis was given. This section gives a more detailed presentation of the various parts of that approach.

As was noted in section 1.2 it is assumed that the decision maker's utility function is

$$U = U(x_1, x_2, \ldots, x_m; u_1, u_2, \ldots, u_n) \tag{3.3.1}$$

where $x_1, x_2, \ldots, x_m$ are attributes that describe the characteristics of the possible outcomes of the decision making process and $u_1, u_2, \ldots, u_n$ are the utilities of the various individuals or groups whose views are important to the decision maker. In order to apply equation (3.3.1) to practical situations it is necessary to

i) determine the actual functional form of U,

ii) assess the utilities u_i, $i = 1, 2, \ldots, n$ of the groups or individuals of interest, and

iii) deal with uncertainties in the values of the x_i's and the u_i's.

In order to determine the functional form of U an approach is taken similar to that of Fleming and Harsanyi discussed in sections 2.2 and 2.4. That is, "reasonable" conditions are imposed on the way in which the decision maker should analyze such situations and, as a result of these, the functional form of U is restricted greatly. Procedures are then devised to complete the specification of U for any particular decision problem.

The assessment of the utilities u_i, $i = 1,2,\ldots,n$ needed to evaluate U is complicated because the preferences of different individuals may be interdependent. That is,

$$u_i = u_i(x_1,x_2,\ldots,x_m;u_1,u_2,\ldots,u_{i=1},u_{i+1},\ldots,u_n). \tag{3.3.2}$$

They might also depend on the decision maker's preferences. That is,

$$u_i = u_i(x_1,x_2,\ldots,x_m;U\ u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_n). \tag{3.3.3}$$

Thus, even if the functional forms of equations (3.3.2) and (3.3.3) were known, it would be necessary to solve a complicated set of interdependent equations to obtain each person's utility for a particular outcome.

However, as will be shown in chapter V, it would be reasonable in many cases to assume that the preferences were not interdependent, i.e.,

$$u_i = u_i(x_1,x_2,\ldots,x_m). \tag{3.3.4}$$

In this case the assessment problem is a standard one of determining a multiattribute utility function. Methods have been developed to do this in some cases.[6] In this thesis procedures are developed to approximately assess the u_i's in more general situations.

In those situations where there is uncertainty in the u_i's due either to possible inaccuracies in the assessed u_i's or to sampling error caused by not assessing all the u_i's, it is necessary to have methods of accounting for the uncertainty. Decision analytic ways of doing this are developed in chapter VI. These usually involve assuming some particular functional form for the u_i's and then assessing probability distributions over unspecified parameters of the functional form. This is, of course, an approximation to the actual situation, but one that is adequate for many practical purposes.

3.4 Summary of Important Results

3.4.1 Chapter IV Results

This chapter considers the problem of determining the specific form of $U(x_1,x_2,\ldots,x_m;u_1,u_2,\ldots,u_n)$. The approach used is to consider reasonable constraints on the preference structure of the decision maker and show how these restrict the form of U. In particular, a number of results of Keeney involving utility independence and preferential independence are applied.[7]

For notational simplicity let $\underline{x} = [x_1,x_2,\ldots,x_m]$ and $\underline{u} = [u_1,u_2,\ldots,u_n]$. It is shown in section 4.1 that often $\underline{x}$ and $\underline{u}$ will be mutually utility independent so that

$$U(\underline{x};\underline{u}) = K_1U_x(\underline{x})+K_2U_u(\underline{u})+K_3U_x(\underline{x})U_u(\underline{u}) \tag{3.4.1}$$

where $U_x(\underline{x})$ and $U_u(\underline{u})$ are conditional utility functions, and K_1, K_2 and K_3 are constants. Furthermore, in section 4.3 it is shown that often the u_i's will be order-one mutually utility independent and have condtional utility functions that are linear in their attributes. This leads to

$$U_u(\underline{u}) = \sum_{i=1}^{n} k_iu_i + \sum_{\substack{i=1\\j>i}}^{n} k_{ij}u_iu_j + \ldots + \lambda u_1u_2\ldots u_n \tag{3.4.2}$$

where k_i, $i = 1,2,\ldots,n, k_{ij}$, $i = 1,2,\ldots,n, j > i, \ldots, \lambda$ are scaling constants.

Section 4.3 also demonstrates that if the u_i's are pair-wise preferentially independent in addition to the conditions that led to equation (3.4.2) then either

$$U_{\underline{u}}(\underline{u}) = K^{-1}\left[\prod_{i=1}^{n} (K k_i u_i + 1) - 1\right]$$

or (3.4.3)

$$U_{\underline{u}}(\underline{u}) = \sum_{i=1}^{n} k_i u_i$$

where $K, k_1, k_2, \ldots, k_n$ are constants.

In section 4.5 procedures are developed to assess the scaling constants needed to completely specify $U(\underline{x};\underline{u})$ in the above equations.

3.4.2 Chapter V Results

In chapter V results are derived that simplify the assessment of the utility functions u_i, $i = 1,2,\ldots,n$. In particular, the case where $u_i = u_i(\underline{x})$ is considered in detail. The idea of parametric dependence is introduced as a way of approximating the utility function in cases where utility independence among the attributes does not hold. An attribute x_j is said to be parametrically dependent on its complement $x_{\bar{j}} = \{x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m\}$ if conditional utility functions over x_j depend on $x_{\bar{j}}$ only through a single parameter $\theta = \theta(x_{\bar{j}})$. That is

$$u_i(\underline{x}) = C_1(x_{\bar{j}}) + C_2(x_{\bar{j}})\, u[x_i \,|\, \theta(x_{\bar{j}})] \tag{3.4.4}$$

where $C_1(x_{\bar{j}})$ and $C_2(x_{\bar{j}})$ are unspecified except that $C_2(x_{\bar{j}}) > 0$, and $u[x_i \,|\, \theta(x_{\bar{j}})]$ is a functional form with $\theta(x_{\bar{j}})$ unspecified. Thus, for example, u might be given by

$$u[x_i \,|\, \theta(x_{\bar{j}})] = -e^{-\theta(x_{\bar{j}})\, x_i} \tag{3.4.5}$$

In section 5.2 it is shown how various combinations of utility independence and parametric dependence simplify the utility assessment problem when u_i depends on two attributes. In section 5.3 these results are generalized to the N attribute case.

3.4.3 Chapter VI Results

Methods are presented that may be used to deal with uncertainty in the u_i's. In cases where the u_i's are utility functions for groups of people rather than individuals it may be reasonable to assume that the u_i's are probabilistically independent of each other. In this case it is only necessary to assess the expected value of each u_i rather than the whole probability distribution for it to specify the decision maker's utility U.

When it is not reasonable to assume that the u_i's are probabilistically independent it is still possible to derive results that are useful in practical applications. The approach taken is to make assumptions about the form of the probability distribution for the u_i's that allow the problem to be structured sufficiently to be analytically tractable. Although the assumptions may not be exactly obeyed in some cases, they should provide useful approximations.

The use of sample data to improve the probability distribution for the u_i's is also considered. It is shown how a sample of the utilities of interest may be used to update the probability distribution for the u_i's.

3.4.4 Chapter VII Results

The three applications given in Chapter VII are interesting by themselves. One studies citizen participation in community decision making; the

second studies time-share computer users' preferences for different computer system characteristics; and the third considers the assessment of the residential preferences of persons being relocated by highway construction.

However, their principal purpose in this thesis is to demonstrate the applicability of the theoretical results of the thesis to practical problems.

The applications demonstrate the strengths and weaknesses of the methods developed here for incorporating the preferences of other individuals or groups into a formal analysis. The principal strength is that a common approach is provided for dealing with a fairly broad class of problems. Until now most approaches for incorporating the preferences of others into a analysis have been ad hoc for a particular problem or a small class of problems.

The principal weakness of the method is that it is necessary to make numerous assumptions and approximations in order to make the analysis tractable. This is a weakness shared by almost all quantitative methods of analysis.

However, as the applications show, the methods are still useful even after the necessary approximations have been made.

Chapter III Footnotes

1. See North[25], Howard[13], Raiffa[28], or Pratt, Raiffa and Schlaifer[27] for a more detailed discussion of decision theory.

2. See the references in footnote 1 for a discussion of these conditions.

3. See Keeney[16, 18] and Raiffa[29].

4. Schlaifer[31] discusses these in detail. In particular, he gives a good introduction to simulation.

5. See Keeney[16, 17, 18, 19], Raiffa[29], and Keeney and Raiffa[20].

6. See Keeney[16].

7. See Keeney[16, 17, 18, 19] for a detailed discussion of utility independence and preferential independence. A brief discussion of these concepts is given in Chapter V.

Chapter IV

UTILITY FUNCTIONS WITH PREFERENCES AS ATTRIBUTES

4.0 Introduction

In order to formally incorporate the preferences of other individuals or groups into a decision analysis, the decision maker must assess the utility function $U(\underline{x};\underline{u})$ where $\underline{x} = [x_1, x_2, \ldots, x_m]$ is the attribute set describing the characteristics of the possible outcomes and $\underline{u} = [u_1, u_2, \ldots, u_n]$ is the vector of utilities of the individuals or groups of interest to the decision maker.

In general, this assessment is difficult since it requires the determination of an (m+n) - dimensional function. However, because $u_1, u_2, \ldots, u_n$ are utility functions a number of simplifying assumptions about th form of U may often be made. These involve various utility independence and preferential independence properties which are reviewed in the next section.

4.0.1 Utility and Preferential Independence

Consider two vector attributes Y and Z. Y is said to be utility independent of Z if the decision maker's preferences over any lotteries on Y for a fixed Z_o in Z are the same regardless of the value of Z_o. That is, if Z_o is the same for all consequences the decision maker's relative preferences for lotteries involving these consequences depend only on Y.[1]

Keeney has shown[2] that if Y and Z are utility independent of each other then

$$u(y,z) = k_1 u_1(y) + k_2 u_2(z) + k_3 u_1(y)\, u_2(z) \qquad (4.0.1)$$

where $u_1(y)$ and $u_2(z)$ are conditional utility functions, and k_1, k_2 and k_3 are scaling constants.

For notational simplicity let

$$y_{\bar{i}} = [y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n] \tag{4.0.2}$$

where y_i, $i = 1, 2, \ldots, n$, is a scalar attribute. The y_i's are said to be <u>order-one mutually utility independent</u> if y_i is utility independent of $y_{\bar{i}}$ for all i. Keeney has shown[3] that if $y_1, y_2, \ldots, y_n$ are order-one mutually utility independent then

$$u(y_1, y_2, \ldots, y_n) = k_o + \sum_{i=1}^{n} k_i u_i(y_i) \tag{4.0.3}$$

$$+ \sum_{\substack{i=1 \\ j>i}} k_{ij} u_i(y_i) u_j(y_j) + \cdots + \lambda u_1(y_1) u_2(y_2) \cdots u_n(y_n)$$

where $u_i(y_i)$, $i = 1, 2, \ldots, n$, is a conditional utility function and the subscripted k's and λ are scaling constants. (There are 2^n of these, two of which are arbitrary.)

For two vector attributes Y and Z, Y is said to be <u>preferentially independent</u> of Z if the decision maker's conditional preference structure in the Y-space for a given z_o in Z is the same regardless of the value of z_o. That is, Y is preferentially independent of Z if the indifference sets in Y for a given z_o in Z do not depend on the value of z_o.[4]

For notational simplicity let

$$y_{\overline{ij}} = [y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n] \quad (4.0.4)$$

where y_i, $i = 1, 2, \ldots, n$, is a scalar attribute. The y_i's are said to be <u>order-two mutually preferentially independent</u> if $\{y_i, y_j\}$ is preferentially independent of $y_{\overline{ij}}$ for all i and j. Keeney has shown[5] that if y_i is utility independent of $y_{\overline{i}}$ for at least one i, and the y_i's are order-two mutually preferentially independent then either

$$u(y_1, y_2, \ldots, y_n) = K^{-1} \left| \prod_{i=1}^{n} [K k_i u_i(y_i) + 1] - 1 \right| \quad (4.0.5)$$

or

$$u(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} k_i u_i(y_i) \quad (4.0.6)$$

where $u_i(y_i)$, $i = 1, 2, \ldots, n$, is a conditional utility function scaled such that the least preferred value of y_i has a utility greater than or equal to zero, and $K, k_1, k_2, \ldots, k_n$ are scaling constants.

4.1 Utility Independence of $\underline{x}$ and $\underline{u}$

Often it is reasonable to assume that $\underline{x}$ and $\underline{u}$ are mutually utility independent for a decision maker wishing to assess $U(\underline{x}; \underline{u})$. The argument is as follows: Consider lotteries over $\underline{x}$ with $\underline{u}$ held fixed at $\underline{u}_o$. The decision maker's relative preferences for different lotteries may not change for different values of $\underline{u}_o$ since the value of $\underline{u}_o$ does not affect the characteristics of the lottery outcomes (which are described by $\underline{x}$). In the same way if $\underline{x}$ is held fixed at $\underline{x}_o$ the decision maker's relative preferences for different lotteries

over $\underline{u}$ may not change if $\underline{x}_o$ is changed since the value of $\underline{x}_o$ has no affect on the preferences of the individuals or groups indicated by $\underline{u}$.

This last statement may seem strange since the preferences $u_1, u_2, \ldots, u_n$ will depend on the value of $\underline{x}$ in most cases. Although this is true it is not relevant to the utility independence argument. When that argument is made it is assumed that the lotteries over $\underline{u}$ are imposed with $\underline{x}$ held fixed. That is, the actual causal mechanisms linking $\underline{x}$ and $\underline{u}$ are conceived to be suspended temporarily and lotteries ignoring these are instituted.

When this fact is realized the utility independence of $\underline{u}$ from $\underline{x}$ seems more reasonable. If $\underline{x}$ and $\underline{u}$ are mutually utility independent then, as noted in section 4.0.1,

$$U(\underline{x};\underline{u}) = K_1 U_x(\underline{x}) + K_2 U_u(\underline{u}) + K_3 U_x(\underline{x}) U_u(\underline{u}) \tag{4.1.1}$$

where $U_x(\underline{x})$ and $U_u(\underline{u})$ are conditional utility functions, and K_1, K_2 and K_3 are scaling constants.

Even if utility independence between $\underline{x}$ and $\underline{u}$ does not seem reasonable to a particular decision maker, Keeney has noted[6] that (4.1.1) provides a very versatile form for approximating $U(\underline{x};\underline{u})$. Thus it will be assumed that (4.1.1) holds for the remainder of this chapter.

Assessing $U(\underline{x};\underline{u})$ using (4.1.1) requires that $U_x(\underline{x})$ and $U_u(\underline{u})$ be determined. Nothing can be said about the form of $U_x(\underline{x})$ without considering a specific problem. Other researchers have shown how the assessment might be done in specific decision problems,[7] and it will not be considered further here.

On the other hand, since the attributes $u_1, u_2, \ldots, u_n$ in $U_u(\underline{u})$ are utility functions there are a number of statements that may be made about its form without having to consider a specific decision problem. The assessment of $U_u(\underline{u})$ will be considered in sections 4.2 - 4.4.

In addition to $U_x(\underline{x})$ and $U_u(\underline{u})$, it is necessary to determine K_1, K_2 and K_3 to determine $U(\underline{x};\underline{u})$ from equation (4.1.1). This problem will be considered in section 4.5.4.

4.2 Symmetry Properties for $U_u(\underline{u})$

In some cases a decision maker would wish his utility $U_u(\underline{u})$ to remain the same if the preferences of various members of the group were interchanged. That is, the identity of the people holding particular views would not influence the manner in which they were taken into account.

There are many situations where the identity of the individuals or groups holding particular views would be important to the decision maker. However, in some of these cases there would be subgroups within which the identity of individuals would not be of interest. For example, a school board considering various plans to end racial imbalance might wish to distinguish whether a person has children in school or not when considering his preferences. However, the particular individuals within each of those groups holding various views might not be of interest.

If the decision maker does not wish to distinguish between the $i^{\underline{th}}$ and $j^{\underline{th}}$ individual's or group's preferences then

$$U_u(u_i, u_j ; u_{\overline{ij}}) = U_u(u_j, u_i ; u_{\overline{ij}}). \tag{4.2.1}$$

That is, U_u is symmetric with respect to u_i and u_j. Symmetry relations like this reduce the region over which U_u must be assessed to be completely specified.

4.2.1 $U_u(\underline{u})$ With Symmetric Attributes

The results given here indicate how symmetry reduces the region where U_u must be assessed.

Result 4.2.1. Suppose $U_u(\underline{u})$ is symmetric with regard to all its attributes. Then $U_u(\underline{u})$ is completely specified if it is known for the region

$$u_1 \leq u_2 \leq \ldots \leq u_{n-1} \leq u_n. \tag{4.2.2}$$

Proof. Since $U_u(\underline{u})$ is symmetric with respect to all its attributes, its functional arguments may be interchanged until the smallest is first, the second smallest is second and so on, and U_u evaluated at the resulting point will be equal to U_u at the initial point. But the resulting point is in the region specified by (4.2.2). Hence the result is proved.

A slight generalization of this is

Result 4.2.2. Suppose $U_u(\underline{u})$ is symmetric with regard to all its attributes. Then it is completely specified if it is known for the region

$$U_{\ell_1} \leq u_{\ell_2} \leq \ldots \leq u_{\ell_{n-1}} \leq u_{\ell_n} \tag{4.2.3}$$

where $\ell_1, \ell_2, \ldots, \ell_n$ is any permutation of $1, 2, \ldots, n$.

The proof of this is straightforward from the proof of result 4.2.1 and can be furnished by the reader.

One final result extends the above reasoning to the case where the views of some, but not all, of the individuals or groups can be interchanged and the decision maker's utility stays the same.

Result 4.2.3. Suppose $U_u(\underline{u})$ is symmetric with regard to the attributes

$$u_{\ell_1}, u_{\ell_2}, \ldots, u_{\ell_m}$$

where $\ell_1, \ell_2, \ldots, \ell_m$ is a subset of $1,2,\ldots,n$. Then U_u is completely specified if it is known for the region

$$u_{\ell_1} \leq u_{\ell_2} \leq \ldots \leq u_{\ell_m} .$$

The proof of this is very similar to that for the other two results.

Results 4.2.1 - 4.2.3 show that imposing indistinguishability on the utilities of various groups or individuals can reduce the region over which $U(\underline{u})$ must be assessed to be completely known. In the next section one particular procedure for assessing utility functions is considered and it is shown how symmetry reduces the labor involved.

4.2.2 Example of Utility Assessment With Symmetric Attributes

One procedure for assessing a utility function $U(u)$ over one attribute is to rescale u so that all feasibe points lie in the interval $0 \leq u \leq 1$, and then to assess the utilities of the points k/m, $k = 1,2,\ldots,m$ and fair a utility curve through the values at these points.

This procedure may be extended to n attributes $[u_1, u_2, \ldots, u_n]$. Each attribute is rescaled so that all feasible values lie between 0 and 1. Then the utilities of the points $[k_1/m, k_2/m, \ldots, k_n/m]$, $k_1, k_2, \ldots, k_n = 1,2,\ldots,m$, are assessed and a utility curve is faired through these values.

If there are n attributes then the utilities of m^n points must be assessed (or actually m^n-2 since the values of two points may be set arbitrarily). However, if the utility function is symmetric in all its attributes then by result 4.2.1 it is only necessary to assess the utilities which lie in the region

$$u_1 \leq u_2 \leq \ldots \leq u_n.$$

It is shown in appendix 4.1 that there are

$$N_s \equiv \binom{m+n-1}{n} = \frac{(m+n-1)!}{n!(m-1)!} \tag{4.2.4}$$

such points. Since the values of two of these points may be set arbitrarily it is necessary to assess the utilities of N_s-2 points.

A comparison of m^n-2 and N_s-2 is given in Figure 4.1 for several values of m and n. This shows that the results of symmetry can be striking. However, if the decision maker wishes to assess his utility over the preferences of more than a few people the problem is formidable even with symmetry. Also, as pointed out above, there are cases when it is not reasonable for $U_u(\underline{u})$ to have symmetric attribut s.

Thus it is necessary to look for other ways in addition to symmetry to simplify the assessment of $U_u(\underline{u})$.

4.3 Utility and Preferential Independence

4.3.1 Order-one Mutual Utility Independence

Consider lotteries that involve uncertainties in the utility u_i of only one individual or group. (That is, all possible outcomes will result in the same value $u_{\bar{i}}$ of everyone else's preferences.) Then many decision makers

m	n	m^n-2	N_s-2	$\frac{N_s-2}{m^n-2}$
3	1	1	1	1.00
	2	8	4	.50
	3	25	8	.32
	4	79	13	.16
	5	241	19	.079
	10	59,047	64	.0008
	50	7.18×10^{23}	1,319	1.8×10^{-21}
5	1	3	3	1.00
	2	23	13	.57
	3	123	33	.27
	4	623	68	.09
	5	3.13×10^{3}	124	.040
	10	9.77×10^{6}	999	1.03×10^{-4}
	50	8.82×10^{34}	3.17×10^{5}	3.59×10^{-30}
10	1	8	8	1.00
	2	98	53	.54
	3	998	218	.22
	4	9,998	713	.071
	5	10^{5}	2,000	.020
	10	10^{10}	92,376	9.24 x 10-6
	50	10^{50}	1.26×10^{10}	1.26×10^{-40}

n = number of attributes.

m^n-2 = number of points whose utility must be assessed without symmetry.

$N_s-2 = \binom{m+n-1}{n} -2$ = number of points whose utility must be assessed with symmetry.

Figure 4.1. Effects of Symmetry on Utility Assessment Problem.

would wish their own preferences to be the same as the preferences of the single person or group affected. That is a conditional utility function over u_i should be proportional to u_i, or

$$U(u_i;u_{\bar{i}}) = c_1(u_{\bar{i}}) + c_2(u_{\bar{i}})\,u_i \qquad (4.3.1)$$

where $c_1(u_{\bar{i}})$ and $c_2(u_{\bar{i}})$ are unspecified functions except that $c_2(u_{\bar{i}})$ is positive.

Often this assumption would be reasonable. However, if the decision maker were worried about having balance among the preferences $u_1, u_2, \ldots, u_n$ then it might not be reasonable to assume that (4.3.1) holds. For example, if the preferences of those people not affected by the lottery were high, the decision maker might prefer high values of u_i while if they were low he might prefer low values of u_i since this would tend to keep the preferences of everyone in the group about the same.

However, if there were a fairly large number of individuals or groups whose views were being taken into account, then variations in the preferences of any one individual or group would not greatly affect the overall pattern of preferences in the group. Thus, even if this pattern were important to the decision maker he might still wish equation (4.3.1) to hold.

If (4.3.1) is true for all i, then the u_i's are order-one mutually utility independent. Furthermore, the conditional utility function over each u_i is linear in u_i. Thus it follows from Keeney's result quoted in section 4.0.1 that

$$U_u(\underline{u}) = k_o + \sum_{i=1}^{n} k_i u_i + \sum_{\substack{i=1 \\ j>i}}^{n} k_{ij} u_i u_j + \ldots + \lambda u_1 u_2 \ldots u_n \qquad (4.3.2)$$

where the subscripted k's and λ are constants.

In this case $U_u(\underline{u})$ will be completely specified if the values of the 2^n-2 scaling constants are established. If, in addition, the utility function is symmetric with respect to the u_i's the number of scaling constants needed is even less as shown by

Result 4.3.1. If

$$U_u(\underline{u}) = k_o + \sum_{i=1}^{n} k_i u_i + \sum_{\substack{i=1 \\ j>i}}^{n} k_{ij} u_i u_j + \ldots + \lambda u_1 u_2 \ldots u_n \qquad (4.3.3)$$

and if it is symmetric with respect to all its attributes, then

$$U_u(\underline{u}) = K_o + K_1 \sum_{i=1}^{n} u_i + K_2 \sum_{\substack{i=1 \\ j>i}}^{n} u_i u_j + \ldots + K_n u_1 u_2 \ldots u_n \qquad (4.3.4)$$

where K_o, K_1, ...K_n are scaling constants.

Proof. Assume without loss of generality that u_i = o is a feasible value of u_i for all i. Also assume for notational convenience that $(u_i;o)$ means that all attributes except u_i equal zero, and similary $(u_i, u_j;o)$ means that all attributes except u_i and u_j equal zero.

From equation (4.3.3) it follows that

$$U(u_i;o) = k_o + k_i u_i \tag{4.3.5}$$

and

$$U(u_j;o) = k_o + k_j u_j. \tag{4.3.6}$$

But, by symmetry,

$$U(u_i, u_j; o) = U(u_j, u_i; o)$$

and hence setting $u_j = u_i$ in equation (4.3.6) and equating (4.3.5) and (4.3.6) yields

$$k_o + k_i u_i = k_o + k_j u_i \tag{4.3.7}$$

which shows that $k_i = k_j$. Since this holds for all i and j, then $k_1 = k_2 = \ldots k_n = K_1$.

I a similar manner it may easily be shown that $k_{ij} = K_2$, $i = 1,2,\ldots,n, j>i$, and so on for the other constants.

Thus with symmetry the number of constants that must be assessed is reduced considerably. There are n + 1 constants in equation (4.3.4). Two can be assigned arbitrarily so that n - 1 must be determined. This contrasts with 2^n-2 when there is no symmetry. The savings can be very substantial as is shown in figure 4.2.

However, the number of constants is still large if the number of individuals or groups whose preferences are to be taken into account is large. Also, of course, it was necessary to assume symmetry in order to derive this result. As can be seen from figure 4.2, if there isn't symmetry the number of scaling constants increases very rapidly. Thus it is useful to investigate possible constraints that will restrict the form of $U_u(\underline{u})$ even more.

n	2^n-2	n-1	$(n-1)/2^n-2)$
1	0	0	-
2	2	1	.50
3	6	2	.33
4	14	3	.21
5	30	4	.13
10	1,022	9	.0088
50	1.13×10^{15}	49	4.3×10^{-14}

Figure 4.2. Effects of Symmetry on the Number of Scaling Constants

4.3.2 Order-two Mutual Preferential Independence

Consider outcomes of the decision making process which differ from each other only in the utilities u_i and u_j of two individuals or groups. Then many decision makers might wish their own preference rankings of the different outcomes to depend only on the utilities u_i and u_j of the two people whose preferences differ and not on the value $u_{\overline{ij}}$ of the other utilities.

Although this seems readonable in many cases, it ignores some questions of balance among the values of the different u_i's just as some of these were ignored when the utility independence conditions of the last section were imposed.

However, if it is accepted for all u_i and u_j, then the u_i's are order-two mutually preferentially independent. If this is true in addition to the order-one mutual utility independence and linearity of the conditional utility functions over the u_i's discussed in the last section, then either

$$U_u(\underline{u}) = K^{-1}\left[\prod_{i=1}^{n}(Kk_i u_i + 1) - 1\right] \qquad (4.3.8)$$

or

$$U_u(\underline{u}) = \sum_{i=1}^{n} k_i u_i \qquad (4.3.9)$$

where $K, k_1, k_2, \ldots, k_n$ are scaling constants.

There are $n + 1$ constants in equation (4.3.8) and n in equation (4.3.9). Since one of these is arbitrary in each case, it is necessary to assess n constants to specify $U_u(\underline{u})$ in equation (4.3.8) and $n-1$ to specify it in (4.3.9).

In those cases where $U_u(\underline{u})$ is symmetric with respect to all its attributes equations (4.3.8) and (4.3.9) reduce to

$$U_u(\underline{u}) = K^{-1}\left[\prod_{i=1}^{n} (K k u_i + 1) - 1\right] \tag{4.3.10}$$

and

$$U_u(\underline{u}) = k \sum_{i=1}^{n} u_i \tag{4.3.11}$$

where K and k are scaling constants. In this case the value of one constant must be assessed if (4.3.10) holds and none if (4.3.11) holds.

Harsanyi showed[9] conditions under which equation (4.3.9) holds. Since this is one special case of the result obtained here it is interesting to see what additional conditions must be imposed to obtain that form rather than the form of equation (4.3.8).

Consider the following two lotteries:

$$L_1 : \langle (u,u;u_{\overline{ij}}) ; (o,o;u_{\overline{ij}}) \rangle$$

and

$$L_2 : \langle (u,o;u_{\overline{ij}}) ; (o,u;u_{\overline{ij}}) \rangle$$

for some i and j where there is a 50-50 chance of either outcome occurring in each lottery and where u and $u_{\overline{ij}}$ are arbitrary but fixed. Then it is easy to verify that if the additive form (4.3.9) holds the decision maker must be indifferent between L_1 and L_2. Otherwise the multiplicative form (4.3.8) holds.

In many cases a decision maker would not be indifferent between L_1 and L_2. In L_1 both individuals always end up with the same utility, however, there is a 50-50 chance that this will be an undesirable value. In L_2, on the other hand, there is always a difference in the utilities received by the two individuals, however, one individual always receives a desirable value. It is not clear that one of these situations would always be more desirable to a decision maker, however, it does seem that he would often perceive a difference between the two cases. If that is true then equation (4.3.8) holds.

4.4 Hierarchical Structuring of $U_u(\underline{u})$

Equations (4.3.10) and (4.3.11) in the last section show how treating the preferences of different individuals or groups symmetrically can reduce the labor needed to assess $U_u(\underline{u})$. However, in many cases decision makers wish to distinguish between the preferences of different individuals or groups. In these situations (4.3.10) and (4.3.11) do not hold.

Sometimes there is partial symmetry. The decision maker can divide the people whose views are of concern to him into several groups whose views he wishes to treat differently. However, he does not care to distinguish between the views of different individuals within the same group.

In this case a hierarchical structuring of $U_u(\underline{u})$ is possible. The decision maker can assemble a utility function for each group assuming symmetry over the preferences of the members of the group since he does not wish to distinguish between them. If the assumptions of the last section

are accepted, the utility function for the i^{th} group will be either

$$U_i = C_i^{-1}\left[\prod_{j=1}^{n_i}(C_i c_i u_{ij} + 1) - 1\right] \tag{4.4.1}$$

or

$$U_i = c_i \sum_{j=1}^{n_i} u_{ij} \tag{4.4.2}$$

where C_i and c_i are scaling constants, and $u_{ij}, j=1,2,\ldots,n_i$ are the utility functions of the members of the i^{th} group.

Then the overall utility function U_u can be written as either

$$U_u = K^{-1}\left[\prod_{i=1}^{n}(K k_i U_i + 1) - 1\right] \tag{4.4.3}$$

or

$$U_u = \sum_{i=1}^{n} k_i U_i \tag{4.4.4}$$

where K, $k_1, k_2, \ldots, k_n$ are scaling constants. Here the k_i's will not be equal since the views of the different groups are to be treated differently.

If this procedure is used,advantage can be taken of whatever symmetry exists. It is not necessary to have the complete symmetry required for equations (4.3.10) and (4.3.11) in the last section.

4.5 Assessment of Scaling Constants

In order to complete the specification of $U(\underline{x};\underline{u})$ a number of scaling constants must be assessed. These include $K, k_1, k_2, \ldots, k_n$ in either

$$U_u(\underline{u}) = K^{-1}\left[\prod_{i=1}^{n}(Kk_i u_i + 1) - 1\right] \qquad (4.5.1)$$

or

$$U_u(\underline{u}) = \sum_{i=1}^{n} k_i u_i, \qquad (4.5.2)$$

or, if the conditions for these equations do not hold, then the subscripted k's and λ must be assessed in

$$U_u(\underline{u}) = k_o + \sum_{i=1}^{n} k_i u_i + \sum_{\substack{i=1 \\ j>i}}^{n} k_{ij} u_i u_j + \ldots + \lambda u_1 u_2 \ldots u_n . \qquad (4.5.3)$$

In addition, K_1, K_2 and K_3 must be assessed in

$$U(\underline{x};\underline{u}) = K_1 U_x(\underline{x}) + K_2 U_2(\underline{u}) + K_3 U_x(\underline{x}) U_u(\underline{u}). \qquad (4.5.4)$$

These assessments are considered in this section.

4.5.1 Assessment of Scaling Constants for $U_u(\underline{u})$ with Complete Symmetry

With complete symmetry equations (4.5.1) and (4.5.2) reduce to

$$U_u(\underline{u}) = K^{-1}\left[\prod_{i=1}^{n}(Kku_i + 1) - 1\right] \qquad (4.5.5)$$

and

$$U_u(\underline{u}) = k \sum_{i=1}^{n} u_i . \qquad (4.5.6)$$

If (4.5.6) holds there is no need to assess any constants since k is arbitrary. In (4.5.5) one constant must be assessed.

Since the scales and origins of the various u_i's are arbitrary it is necessary to pin these down so that the decision maker knows what he is comparing when he considers tradeoffs between the different u_i's. One way to do this is to pick values $\underline{x}_i^o$ and $\underline{x}_i^*$ where $\underline{x}_i^o \prec \underline{x}_i^*$ such that the decision maker feels the utility of $\underline{x}_i^o$ is the same to individual i as the utility of $\underline{x}_j^o$ is to the individual j for all i and j. Similarly he assumes $\underline{x}_i^*$ and $\underline{x}_j^*$ have the same utilities for the different individuals.

This is the interpersonal comparison of preferences that, as was pointed out in section 1.1.1, must always be made in any procedure for combining preferences of different people. It will be seen in what follows that these are the only interpersonal comparisons that must be made.

Since the scale and origin of each u_i are arbitrary, $u_i(\underline{x}_i^o)$ and $u_i(\underline{x}_i^*)$ can be given any values. Assume $u_i(\underline{x}_j^o) = 0$ and $u_i(\underline{x}_i^*) = 1$ for convenience. Also assume without loss of generality that $U_u(1, 1, \ldots, 1) = 1$. This implies

$$U_u(\underline{u}) = \frac{\prod_{i=1}^{n} (c u_i + 1) - 1}{(c+1)^n - 1} \tag{4.5.7}$$

where $c = Kk$.

Now c must be assessed. Consider first the case where $n = 2$. Then

$$U(u_1, u_2) = \frac{(cu_1 + 1)(c u_2 + 1) - 1}{(c+1)^2 - 1} . \tag{4.5.8}$$

Since the decision maker has assigned a concrete meaning in terms of the outcomes $\underline{x}$ only to the values 0 and 1 of u_1 and u_2 it seems reasonable to use

only these values in the assessment. Otherwise he will be forced to make more interpersonal comparisons of preferences.

One way to do this is to consider the lotteries

$$L_1 \; : \; \langle u_1 = 1 \text{ and } u_2 = 0 \text{ for sure} \rangle$$

and

$$L_2 \; : \; \langle u_1 = 1,\ u_2 = 1 \ ; \ p \ ; \ u_1 = 0,\ u_2 = 0 \rangle$$

where there is a probability p of obtaining the outcome $u_1 = 1$, $u_2 = 1$ in lottery L_2. The utilities of these two lotteries are

$$U(L_1) = \frac{c}{(c+1)^2 - 1} \tag{4.5.9}$$

and

$$U(L_2) = p. \tag{4.5.10}$$

If the decision maker picks the p such that he is indifferent between L_1 and L_2 then

$$\frac{c}{(c+1)^2 - 1} = p. \tag{4.5.11}$$

This implies that

$$c = \frac{1-2p}{p}. \tag{4.5.12}$$

Thus the value of c is determined.

The case where $n > 2$ can be handled in almost the same way. Consider the lotteries

$$L_3 \; : \; \langle u_1 = 1,\, u_2 = 0,\, u_{\overline{12}} = u^o_{\overline{12}} \text{ for sure}\rangle$$

and

$$L_4 \; : \; \langle u_1 = 1,\, u_2 = 1,\, u_{\overline{12}} = u^o_{\overline{12}} \,;\, p \,;\, u_1 = 0, u_2 = 0, u_{\overline{12}} = u^o_{\overline{12}}\rangle$$

That is, only the utilities u_1 and u_2 vary among the different outcomes. The others are fixed at the values $u^o_3, u^o_4, \ldots, u^o_n$. Then

$$U(L_3) = \frac{(c+1) \prod_{i=3}^{n} (c u^o_i + 1) - 1}{(c+1)^n - 1} \tag{4.5.13}$$

and

$$U(L_4) = p \, \frac{(c+1)^2 \prod_{i=3}^{n} (c u^o_i + 1) - 1}{(c+1)^n - 1} + (1-p) \, \frac{\prod_{i=3}^{n} (c u^o_i + 1) - 1}{(c+1)^n - 1} \; . \tag{4.5.14}$$

If the decision maker picks the p such that he is indifferent between L_3 and L_4 then (4.5.13) and (4.5.14) may be equated. This yields

$$c = \frac{1-2p}{p} \tag{4.5.15}$$

just as in the case where $n = 2$.

The question that the decision maker must answer to assess c is not easy. This type of analysis is not one that most people are used to. Thus they may not be sure that their response represents their true preferences. In order to see whether the exact value of c is very important the sensitivity of $U_u(\underline{u})$ to variations in c is investigated in the next section.

4.5.2 Sensitivity of $U_u(\underline{u})$ to Variations in c

As was shown in the last section

$$U_u(\underline{u}) = \frac{\prod_{i=1}^{n} (c u_2 + 1) - 1}{(c+1)^n - 1} \quad . \tag{4.5.16}$$

In order to see how this is affected by variations in c its values will be plotted for different values of c along the line $u_1 = u_2 = \ldots = u_n = u$ between $u = 0$ and $u = 1$.

Along this line

$$U_u(\underline{u}) = \frac{(cu+1)^n - 1}{(c+1)^n - 1} \; . \tag{4.5.17}$$

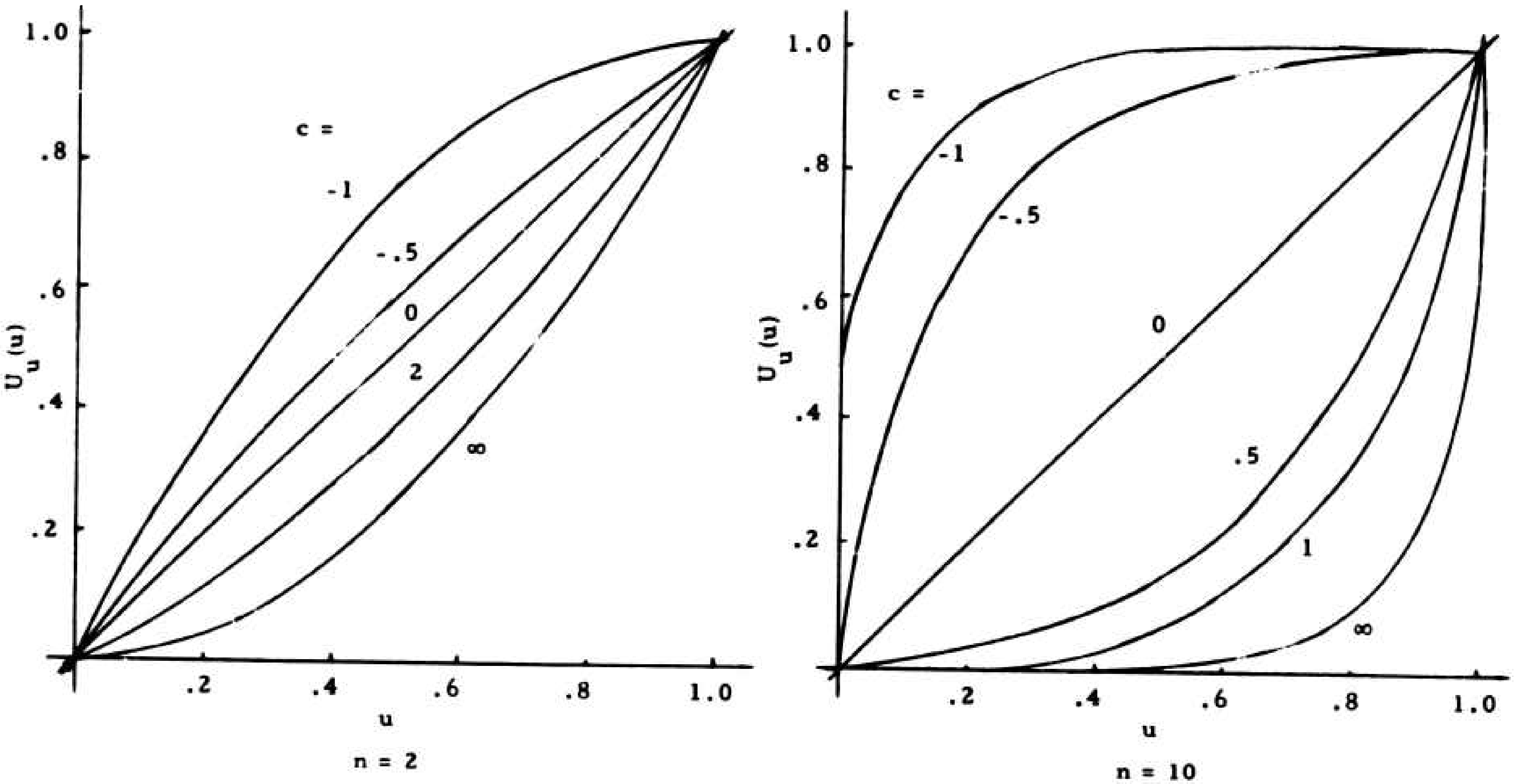
This is plotted in figure 4.3 for n = 2 and n = 10 and for several values of c. Beneath the plots the value of p in equations (4.5.12) and (4.5.15) that corresponds to each c is given. From these plots it can be seen that the amount of variation in $U_u(\underline{u})$ due to changes in c increases as n increases. For n = 10 the variation is quite substantial.

These plots point out the importance of doing sensitivity analyses in any application of this material to see how variations in c affect the results of the decision analysis.

4.5.3 Assessment of Scaling Constants for $U_u(\underline{u})$ with Non-symmetric Attributes

In this section the assessment of the scaling constants $K, k_1, k_2, \ldots, k_n$ in

$$U_u(\underline{u}) = K^{-1} \left[\prod_{i=1}^{n} (K k_i u_i + 1) - 1 \right] \tag{4.5.18}$$



c	-1	-.5	0	.5	1	∞
p	1	2/3	1/2	2/5	1/3	0

Figure 4.3. Plots of $U_u(u)$

and

$$U_u(\underline{u}) = \sum_{i=1}^{n} k_i u_i$$

will be considered. The specific case where $n = 3$ and (4.5.18) holds will be studied. The methods developed for this case can easily be extended to situations where $n \neq 3$ or where (4.5.18) holds.

It is assumed that $\underline{x}_i^o$ and $\underline{x}_i^*$ have been selected in the same manner as in section 4.5.1 and that $u_i(\underline{x}_i^o) = 0$ and $u_i(\underline{x}_i^*) = 1$ for all i. Assume also, without loss of generality, that $U_u(1, 1, \ldots, 1) = 1$. Then (4.5.18) may be rewritten

$$U_u(\underline{u}) = \frac{\prod_{i=1}^{n} (c_i u_i + 1) - 1}{\prod_{i=1}^{n} (c_i + 1) - 1} \tag{4.5.19}$$

where $c_i = Kk_i$, $i = 1, 2, \ldots, n$.

As in section 4.5.1, it seems reasonable to restrict the questions used to assess the c_i's to ones involving $u_i = 0$ or 1 for all i since the decision maker has carefully studied the meaning of the outcomes $\underline{x}_i^o$ and $\underline{x}_i$ that go with these.

For the case where $n = 3$ consider the three vectors of utilities $\underline{u}_1 = (u_1, u_2, u_3) = (1, 0, 0)$, $\underline{u}_2 = (0, 1, 0)$, $\underline{u}_3 = (0, 0, 1)$ and rank them according to preference.

Suppose they rank $\underline{u}_3 \succ \underline{u}_2 \succ \underline{u}_1$. Then from (4.5.19) it must be true that $c_1 < c_2 < c_3$. Now consider the lotteries

$$L_1 \; : \; \langle (1,0,u) \text{ for sure} \rangle$$

and

$$L_2 \; : \; \langle (0,1,u) \; ; \; p_1 ; \; (0,0,u) \rangle$$

where u is arbitrary and L_2 has a probability p_1 of yielding $(0,1,u)$. If p_1 is selected such that L_1 is indifferent to L_2 it follows from (4.5.19) that

$$c_1 = p_1 c_2 . \tag{4.5.20}$$

In the same way find p_2 such that L_3 is indifferent to L_4 where

$$L_3 \; : \; \langle (1,u,o) \text{ for sure} \rangle$$

and

$$L_4 \; : \; \langle (0,u,1) \; ; \; p_2 \; ; \; (o,u,o) \rangle$$

From (4.5.19) it follows that

$$c_1 = p_3 (c_2 c_3 + c_2 + c_3) . \tag{4.5.21}$$

Equations (4.5.19), (4.5.20) and (4.5.21) may be solved to yield

$$c_1 = \frac{p_1 p_2}{p_3} - (p_1 + p_2),$$

$$c_2 = \frac{p_2}{p_3} - (1 + \frac{p_2}{p_1}) \tag{4.5.22}$$

and

$$c_3 = \frac{p_1}{p_3} - (1 + \frac{p_1}{p_2}) .$$

The questions that must be answered to obtain the c_i's are not easy ones. Also, as was shown in the last section, the form of $U_u(\underline{u})$ may be influenced significantly by changes in the c_i's. Thus it will be necessary to carefully check the assessments used to determine the constants and also to check the final result of the decision analysis to see how it is affected by changes in the c_i's.

4.5.4 Assessment of Scaling Constants for $U(\underline{u};\underline{x})$

Once $U_u(\underline{u})$ and $U_x(\underline{x})$ have been determined the constants K_1, K_2 and K_3 in

$$U(\underline{x};\underline{u}) = K_1 U_x(\underline{x}) + K_2 U_u(\underline{u}) + K_3 U_x(\underline{x}) U_u(\underline{u}) \tag{4.5.23}$$

must be assessed to specify $U(\underline{x};\underline{u})$.

Pick two values of $\underline{x}$, $\underline{x}^o$ and $\underline{x}^*$, with $\underline{x}^o \prec \underline{x}^*$, that have been carefully thought about and can be used to compare the decision maker's preferences directly for outcomes, represented by $U_x(\underline{x})$, with his preferences for following the views of other individuals or groups, represented by $U_u(\underline{u})$.

Assume that $U_u(\underline{u})$ has been scaled in the manner of sections 4.5.1 and 4.5.3. Further assume that $U_x(\underline{x})$ has been scaled so that $U_x(\underline{x}^o) = 0$ and $U_x(\underline{x}^*) = 1$.

If $U(\underline{x};\underline{u})$ is scaled so that $U(\underline{x}^o;\underline{0}) = 0$ and $U(\underline{x}^*;\underline{1}) = 1$, where $\underline{0} = [0,0,\ldots,0]$ and $\underline{1} = [1,1,\ldots,1]$, then it follows from (4.5.23) that

$$1 = K_1 + K_2 + K_3. \tag{4.5.24}$$

Now compare $(\underline{x}^* ; \underline{0})$ and $(\underline{x}^o ; \underline{1})$. Suppose, for example, that $(\underline{x}^* ; \underline{0}) \succ (\underline{x}^o ; \underline{1})$. Then, from (4.5.23), it follows that $K_1 > K_2$. Now find the p_1 such that L_1 is indifferent to L_2 where

$$L_1 \ : \ \langle (\underline{x}^o ; \underline{1}) \text{ for sure} \rangle$$

and

$$L_2 \ : \ \langle (\underline{x}^* ; \underline{0}) \ ; \ p_1 \ ; \ (\underline{x}^o ; \underline{0}) \rangle$$

It follows from (4.5.23) that

$$K_2 = p_1 K_1 . \tag{4.5.25}$$

Determine the p_2 such that L_3 is indifferent to L_4 where

$$L_3 \ : \ \langle (\underline{x}^* ; \underline{0}) \text{ for sure} \rangle$$

and

$$L_4 \ : \ \langle (\underline{x}^* ; \underline{1}) \ ; \ p_2 \ ; \ (\underline{x}^o ; \underline{0}) \rangle$$

Then, from (4.5.23) it follows that

$$K_1 = p_2 (K_1 + K_2 + K_3) . \tag{4.5.26}$$

Solving (4.5.24), (4.5.25) and (4.5.26) yields

$$K_1 = p_2,$$

$$K_2 = p_1 p_2, \tag{4.5.27}$$

and

$$K_3 = 1 - p_2 - p_1 p_2 .$$

This completes the determination of the scaling constants K_1, K_2 and K_3.

4.6 Concluding Remarks

This chapter has considered methods of determining the utility function $U(\underline{x};\underline{u})$. The approach taken was to show how certain restrictions on the decision maker's reasoning constrain the form of $U(\underline{x};\underline{u})$. Methods were then developed to completely specify the form of $U(\underline{x};\underline{u})$ in any particular decision problem.

In particular, ways of assessing a number of different scaling constants that arose during the development were given.

Appendix 4.1

In this appendix we derive the result used in section 4.2.2. The result is presented as a theorem.

Theorem A.4.2. Consider the set of all points X in R^n such that

$$x = (x_1, x_2, \ldots, x_n) = \left(\frac{k_1}{m}, \frac{k_2}{m}, \ldots, \frac{k_n}{m} \right)$$

where $k_i = 1, 2, \ldots, m$ for all i. Then the number of these points N_s which meets the condition

$$x_1 \leq x_2 \leq \ldots \leq x_n \tag{A.4.1}$$

is

$$N_s = \binom{m+n-1}{n} = \frac{[m+(n-1)]!}{n!\,(M-1)!} \quad . \tag{A.4.2}$$

Proof. We establish this result by induction on n.

n = 1. Clearly the result is true since all the points must be included in one dimension.

n > 1. We proceed by assuming the result to be true for (n - 1) dimensions and then showing it is true for n dimensions. We do this by considering each possible value of X_n and, using the result assumed to be true for (n - 1) dimensions, find the number of points for that value of X_n which meet condition A.4.1. We then sum the results for each X_n to get the total number of poin's.

Consider $X_n = m/m$. This places no restriction on X_{n-1} and hence, using A.4.2, there are

$$\frac{[m+(n-2)]!}{(n-1)!\,(m-1)!}$$

poin's in the allowed region with $X_n = m/m$.

Consider $X_n = (m-1)/m$. Then X_n is restricted to the values $1/m, 2/m, \ldots, (m-1)/m$. Applying A.4.2 for (n-1) dimensions and (m-1) allowed values gives

$$\frac{[m+(n-2)]!}{(n-1)!\,(m-2)!}$$

points in the allowed region with $X_n = (m-1)/m$.

We can continue the procedure above with $V_n = (m-2)/m$, etc. Summing the results gives

$$N_s = \frac{[m+(n-2)]!}{(n-1)!\,(m-1)!} + \frac{[(m-1)+(n-2)]!}{(n-1)!\,(m-2)!}$$

$$+ \frac{[(m-2)+(n-2)]}{(n-1)!\,(m-3)!} + \ldots + 1 \qquad \text{(A.4.3)}$$

This may be rewritten as

$$N_s = \frac{[m+(n-2)]!}{(n-1)!\,(m-1)!}\left| 1 + \frac{m-1}{m+(n-2)}\right| 1$$

$$+ \frac{m-2}{m+(n-3)}\left\{ 1 + \ldots \frac{k}{n+k-1}\left\{ 1 + \ldots\right.\right.$$

$$\ldots \frac{3}{n+2}\left\{ 1 + \frac{2}{n+1}\left\{ 1 + \frac{1}{n}\right\}\right\} \ldots \Big| \ldots \Big|\Big|\Big| \qquad \text{(A.4.4)}$$

We now evaluate (A.4.4) by induction, Define

$$f_k = 1 + \frac{k}{n+k-1}\left\{ 1 + \ldots \frac{3}{n+2}\left\{ 1 + \frac{2}{n+1}\left\{ 1 + \frac{1}{n}\right\}\right\} \ldots \right\} \qquad \text{(A.4.5)}$$

$\underline{k = 1}$. By inspection $f_k = \frac{n+k}{n}$ for $k = 1$.

$\underline{k > 1}$. $f_k = 1 + \frac{k}{n+k-1} f_{k-1}$. Assume $f_{k-1} = (n+k-1)/n$. Then $f_k = (n+k)/n$.

It follows from this and (A.4.4) that

$$N_s = \frac{[m+(n-2)]!}{(n-1)!\,(m-1)!} \times \frac{n+m-1}{n} \tag{A4.6}$$

$$= \frac{[m+(n-1)]!}{n!\,(m-1)!} .$$

The theorem is thus established.

Chapter IV Footnotes

1. Utility independence is discussed further in Keeney[16].

2. See Keeney[18].

3. See Keeney[18[and Keeney and Raiffa[19].

4. See Raiffa[29] and Keeney and Raiffa[19].

5. See Keeney[19].

6. See Keeney[17].

7. See Keeney[16].

8. This result was discussed in section 4.0.1.

9. See section 2.4.2 or Harsanyi[11].

Chapter V

ASSESSING UTILITY FUNCTIONS FOR GROUP MEMBERS

In the last chapter it was shown that if certain constraints are imposed on the form of $U(\underline{x};\underline{u})$ than its assessment can be broken into three parts:

i) assess a conditional utility function $U_x(\underline{x})$,

ii) determine a number of scaling constants, and

iii) assess the utility functions $u_1, u_2, \ldots, u_n$.

Items i and ii were discussed in chapter IV. In this chapter the assessment of the u_i's is considered.

Although these are standard utility functions they have several features that make the assessment problem different than usual. First, in general it is to be expected that the utility function u_i of the $i^{\underline{th}}$ individual or group depends on the preferences of the decision maker and the other members of the group as well as on the outcomes described by $\underline{x}$. That is,

$$u_i = u_i(\underline{x};U;u_{\overline{i}}) \tag{5.0.1}$$

where $u_{\overline{i}} = [u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n]$. If this is true for all the u_i's then it will be difficult to find the utilities u_i, $i = 1, 2, \ldots, n$ that correspond to each outcome $\underline{x}$ since a set of interdependent equations

$$\begin{aligned} U &= U(\underline{x};\underline{u}) \\ u_1 &= u_1(\underline{x};U;u_{\overline{1}}) \\ u_2 &= u_2(\underline{x};U;u_{\overline{2}}), \\ &\;\vdots \\ u_n &= u_n(\underline{x};U;u_{\overline{n}}) \end{aligned} \tag{5.0.2}$$

will have to be solved.

In some cases it would be reasonable to assume that the utilities depend only on the outcomes $\underline{x}$. That is, $u_i = u_i(\underline{x})$, $i = 1,2,\ldots,n$. This means that the people are not interested in the preferences of the other individuals or groups, or they are informally taking them into account when they assess their utility function over $\underline{x}$.

Even if $u_i = u_i(\underline{x})$, there are several assessment difficulties that remain. First, if the decision maker wishes to take into account the preferences of many people, there may not be time or resources enough to assess all their utility functions. Second, even if their utilities can be assessed, the functions obtained may not represent the views of the individuals accurately. This may be due to a deliberate attempt to conceal true preferences or because they haven't thought carefully enough about what their preferences are. This problem may be particularly acute if many utilities are to be obtained. In that case it becomes difficult to spend the time with each person needed to properly assess his utility function.[1]

In the next chapter methods are developed for dealing with uncertainty due to factors like failure to assess everyone's utility function or uncertain bias in the assessed function. In this chapter methods are developed that help assess utility functions quickly and, at the same time, make it easier to check whether the assessed functions represent the individuals' preferences correctly. Methods of this sort are necessary if the decision analytic approach to incorporating the preferences of others into a formal analysis is to be useful in practical situations.

The approach taken is to assume that the u_i's are functions only of the outcomes $\underline{x}$, and then develop ways to quickly approximate $u_i(\underline{x})$. The approximation method extends work done by other researchers for single attribute utility functions[2] to multiattribute functions. These researchers have identified properties that the utility functions of many real-world decision makers would be expected to have, and then found particular functional forms $u(x) = u(x|\theta_1, \theta_2, \ldots, \theta_m)$ which have these properties and also have one or more arbitrary parameters $\theta_1, \theta_2, \ldots, \theta_m$. Questions are asked of an individual to obtain the values of these parameters and the resulting function is assumed to be his utility function. Thus, for example, it might be assumed that $u(x) = e^{-\theta x}$ and questions would be asked to determine θ.

In general, the utility function obtained this way will only approximate the person's true preferences. However, if the functional form is carefully selected the approximation should be good. Furthermore, people are often uncertain enough about their preferences so that they will be willing to use the function as if it represents their preferences.

In the last section of this chapter the general case where $u_i = u_i(\underline{x}; U; u_{\bar{i}})$ is considered and a simple case is investigated to show how the interdependence may affect the decision analysis.

5.1 Parameterized Functional Forms for Single Attribute Utility Functions

If $u(x)$ is assumed to be of a particular parameterized form $u(x|\theta_1, \theta_2, \ldots, \theta_m)$ then only a few questions need be asked to specify $\theta_1, \theta_2, \ldots, \theta_m$. Therefore, time and effort can be put into making sure that

the answers to these questions represent the preferences of the decision maker correctly.

If a utility function is assessed without assuming a functional form, it is necessary to assess the utilities of many points to obtain an accurate idea of the shape of the shape of the function. This is time consuming and often doesn't leave time for carefully checking to see if the individual's assessments represent his true preferences.

Of course, when a particular functional form is specified for u(x) the possible shapes of the function are limited. Thus the form should be carefully selected so that it can yield a wide variety of possible shapes while still only having a small number of parameters to be determined.

One way to do this is to specify desirable properties for utility functions and then find classes of functions that have these properties. One such set of desirable properties involves risk-aversion.

5.1.1 Risk Aversion

Suppose u(x) is strictly increasing and there is a lottery over x with expected value $\bar{x}$ and variance σ_x^2. Suppose further that σ_x^2 is sufficiently small that the first few terms of the Taylor expansion about $\bar{x}$ are an adequate representation of u(x) over the region where x has significant probability of occurring. Then Pratt has shown[3] that

$$\pi \equiv \bar{x} - x_c \approx -\frac{1}{2} \frac{u''(\bar{x})}{u'(\bar{x})} \sigma_x^2 \qquad (5.1.1)$$

where π is the decision maker's risk premium for the lottery and x_c is his certainty equivalent for it.

Notice that π is proportional to

$$r(\bar{x}) \equiv -\frac{u''(\bar{x})}{u'(\bar{x})} . \tag{5.1.2}$$

This called the risk-aversion function since it indicates how large a risk premium the decision maker is willing to pay to eliminate the uncertainty in the situation he faces.

If the expected value $\bar{x}$ of the lottery is changed to a new value $\bar{x}'$ while σ_x^2 stays the same then often π, and hence $r(\bar{x}')$, will change. For example, if $\bar{x}' > \bar{x}$ then π might decrease since there is less chance that an undesirable value will occur. Thus the decision maker is more nearly willing to use expected value as a guide to decision making since, "on the average," he will receive this amount and he isn't as worried about bad outcomes due to uncertainty wiping him out in the meantime as he was when $\bar{x}$ was smaller.

5.1.2 Constant Risk Aversion

Sometimes the risk premium of a lottery would remain fixed as $\bar{x}$ varies over some region. In this case it follows from (5.1.1) and (5.1.2) that

$$r = \frac{u''(x)}{u'(x)} \tag{5.1.3}$$

where r is a constant. This can easily be solved to yield

$$u(x) = \begin{cases} A - (\operatorname{sgn} r)\, B e^{-rx}, & r \neq 0 \\ A + Bx, & r = 0 \end{cases} \tag{5.1.4}$$

for $u'(x) > 0$, where sgn r is the algebraic sign of r and A and B are unspecified except that $B > 0$.

Thus, if a decision maker wishes his risk-aversion to be constant then his utility function is specified once the value of the parameter r is known. Hence, his utility for only one lottery must be assessed in order to completely specify $u(x)$.

5.1.3 Using the Exponential Utility Function as an Approximation

Howard notes[4] that exponential utility functions serve as adequate approximations to many utility function:

> "The utility functions assessed by actual decision makers...are usually smooth functions that are concave downward and representable by an exponential at least over a limited range of monetary outcomes."

The versatility of the exponential is shown in figure 5.1 where

$$u(x|r) = \frac{1 - e^{-rx}}{1 - e^{-r}} \tag{5.1.5}$$

is plotted for several values of r.

Howard uses the exponential utility function extensively in his methodology for approximately analyzing the effects of uncertainty in large decision problems. He comments that even when the exponential utility function is not a good approximation

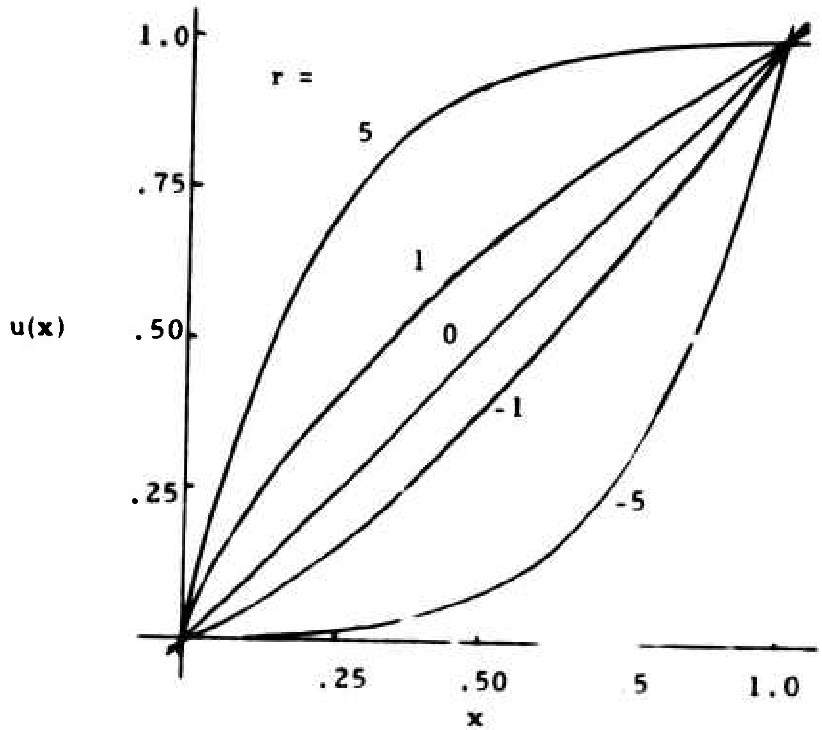


Figure 5.1. Plots of $u(x) = (1 - e^{-rx}) / (1 - e^{-r})$

> the utility function can still be bounded by exponential utility functions having risk aversion coefficients that are the maximum and minimum values of risk aversion coefficient assumed by the actual utility function over the same range. The certain equivalents developed for these exponential utility functions will bound the certain equivalent for the actual utility functions over this range.

In cases where the exponential utility function is not appropriate then other functional forms might be used. For example, Kaufman[5] and Spetzler[6] have investigated the logarithmic utility function $u(x) = A + B \log (x + c)$ for $x > -c$ where A, B and c are constants with $B > 0$. This function has $r(x) = (x+c)^{-1}$ and hence is decreasingly risk averse--a property which would be desirable for some utility functions.

Both the exponential and logarithmic utility functions have one free parameter. That is,

$$u(x) = u(x|\theta) \tag{5.1.6}$$

where θ is the parameter whose value is unspecified. This might be assessed as follows: Pick $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ such that $x^{(1)} \prec x^{(2)} \prec x^{(3)}$. Normalize $u(x|\theta)$ so that $u(x^{(1)}|\theta) = 0$ and $u(x^{(3)}|\theta) = 1$ for all θ. (This is always possible since the scale and origin of a utility function are arbitrary.) Consider the lotteries

$$L_1 : \langle x^{(2)} \text{ for sure} \rangle$$

and (5.1.7)

$$L_2 : \langle x^{(3)} : p : x^{(1)} \rangle$$

where there is a probability p of obtaining $x^{(3)}$ in L_2. Determine the p such that L_1 is indifferent to L_2. Then it follows from (5.1.7) that

$$p = u(x^{(2)} \mid \theta). \tag{5.1.8}$$

This can be solved for θ and hence u(x) is completely specified.

5.2 Parameterized Functional Forms for Two Attribute Utility Functions

In this section the approach to utility assessment discussed in the last section is extended to the two attribute case.

Keeney has discussed[7] situations where one attribute x is utility independent of another. That is, the conditional utility function $u_x(x)$ for any fixed y is the same. In this case

$$u(x,y) = c_1(y) + c_2(y)\, u_x(x) \tag{5.2.1}$$

where $c_1(y)$ and $c_2(y)$ are unspecified except $c_2(y) > 0$.

He showed[8] how utility independence reduces the assessment necessary to specify u(x,y). Unfortunately, in many cases of practical interest utility independence does not hold.

However, it might often be adequate to assume that any conditional utility function over x for a fixed y could be selected from the parameterized family $u_x(x|\theta)$ with the parameter varying depending on the value of y. In this case

$$u(x,y) = c_1(y) + c_2(y)\, u_x[x \mid \theta(y)]. \tag{5.2.2}$$

This form is fairly general.

For example if $u_x(x|\theta$ were the exponential form discussed in section 5.1 then the conditional utility functions over x could be any of the curves

shown in figure 5.1 depending on the value of θ.

If (5.2.2) holds then x will be referred to as parametrically dependent on y. This terminology is used because conditional utility functions over x depend on y only through the parameter θ.

In some cases (5.2.2) will hold only for certain values of y. In this case x will be referred to as parametrically dependent on y for $y_1, y_2, \ldots, y_r$ where $y_1, y_2, \ldots, y_r$ are the values for which (5.2.2) holds.

Parametric dependence might be reasonable in many cases where utility independence was not. As will be seen in the next three sections, parametric dependence conditions reduce greatly the amount of data needed to assess a utility function $u(x,y)$.

5.2.1 Parametric Dependence and Utility Independence

In the derivations of this section and those that follow it is assumed that for every a and x there exists a unique θ such that $a = f_x(x|\theta)$.

Keeney showed[9] that if x and y are mutually utility independent then $u(x,y)$ is determined by two conditional utility function $u_x(x)$ and $u_y(y)$, and the utilities of any two of the four points $(x^{(i)}, y^{(j)})$, $i, j = 1, 2$. A similar result is now proved for the case where x is utility independent of y, but y is parametrically dependent on x.

Theorem 5.2.1. Suppose x is utility independent of y and y is parametrically dependent on x for $x^{(1)}$ and $x^{(2)}$. Then $u(x,y)$ is completely specified by a conditional utility function $u_x(x)$, the parametric form $u_y(y|\theta)$ and the

utilities of any four of the points $(x^{(i)}, y^{(j)})$, $i = 1,2$, $y = 1,2,3$, (where $x^{(1)} \prec x^{(2)}$ and $y^{(1)} \prec y^{(2)} \prec y^{(3)}$).

Proof. Assume, without loss of generality, that $u_x(x^{(1)}) = u_y(y^{(1)}|\theta) = 0$ and $u_x(x^{(2)}) = u_y(y^{(3)}|\theta) = 1$.

Since y is parametrically dependent on x for $x^{(1)}$ and $x^{(2)}$, then

$$u(x^{(i)}, y) = d_1(x^{(i)}) + d_2(x^{(i)}) u_y[y|\theta(x^{(i)})] \text{ for } i = 1,2. \tag{5.2.3}$$

Further, since x is utility independent of y,

$$u(x,y) = c_1(y) + c_2(y)\, u_x(x). \tag{5.2.4}$$

Form (5.2.3) it follows that

$$\begin{aligned} u(x^{(i)}, y) &= u(x^{(i)}, y^{(1)}) \\ &\quad + [\,u(x^{(i)}, y^{(3)}) - u(x^{(i)}, y^{(1)})]\, u_y[y|\theta(x^{(i)})] \end{aligned} \tag{5.2.5}$$

for $i = 1,2$. Thus $u(x^{(i)}, y)$ would be known if $\theta(x^{(i)})$ were known. Set $y = y^{(2)}$ in (5.2.5). Then

$$u_y[y^{(2)}|\theta(x^{(i)})] = \frac{u(x^{(i)}, y^{(2)}) - u(x^{(i)}, y^{(1)})}{u(x^{(i)}, y^{(3)}) - u(x^{(i)}, y^{(1)})} . \tag{5.2.6}$$

This may be solved for $\theta(x^{(i)})$.

From (5.2.4) it follows that

$$u(x,y) = u(x^{(1)}, y) + [u(x^{(2)}, y) - u(x^{(1)}, y)]\, u_x(x). \tag{5.2.7}$$

But $u(x^{(1)}, y)$ and $u(x^{(2)}, y)$ are known from (5.2.5) and (5.2.6). Hence $u(x,y)$ is determined.

Although all six of the points $u(x^{(i)}, y^{(j)})$, $i = 1,2$, $j = 1,2,3$ have been used in this proof, two of them may be specified arbitrarily since the scale and origin of $u(x,y)$ are arbitrary.

* * * * * * *

This theorem shows that if one attribute is parametrically dependent on the other then the utilities of only two more points must be assessed to determine the utility function than in the case where the attribute is utility independent.

5.2.2 Mutual Parametric Dependence

In this section the case where neither attribute is utility independent of the other is considered, but where there is parametric dependence between them. The theorem below proves that, in addition to the parametric functionals $u_x(x|\theta)$ and $u_y(y|\theta)$, it is only necessary to have the utilities of seven points to specify $u(x,y)$.

Theorem 5.2.2. Suppose x is parametrically dependent on y and y is parametrically dependent on x for $x^{(1)}, x^{(2)}$ and $x^{(3)}$. Then $u(x,y)$ is determined by the parametric forms $u_x(x|\theta)$, $u_y(y|\theta)$ and the utilities of seven of the points $(x^{(i)}, y^{(j)})$, $i, j = 1,2,3$ (where $x^{(1)} < x^{(2)} < x^{(3)}$ and $y^{(1)} < y^{(2)} < y^{(3)}$).

Proof. Assume, without loss of generality, that $f_x(x^{(1)}|\phi) = f_y(y^{(1)}|\theta) = 0$ and $f_x(x^{(3)}|\phi) = f_y(y^{(3)}|\theta) = 1$.

From the conditions of the theorem

$$u(x,y) = d_1(y) + d_2(y)\, u_x[x|\phi(y)] \tag{5.2.8}$$

and

$$u(x^{(i)},y) = c_1(x^{(i)}) + c_2(x^{(i)})\, u_y[y \mid \theta(x^{(i)})] \tag{5.2.9}$$

for $i = 1,2,3$. It follows from these equations that

$$u(x,y) = u(x^{(1)},y) + [u(x^{(3)},y) - u(x^{(1)},y)]\, u_x[x \mid \phi(y)] \tag{5.2.10}$$

and

$$u(x^{(i)},y) = u(x^{(i)},y^{(1)}) + [u(x^{(i)},y^{(3)}) - u(x^{(i)},y^{(1)})] \times u_y[y \mid \theta(x^{(i)})]. \tag{5.2.11}$$

Setting $y = y^{(2)}$ in (5.2.11) yields

$$u_y[y^{(2)} \mid \theta(x^{(i)})] = \frac{u(x^{(i)},y^{(2)}) - u(x^{(i)},y^{(1)})}{u(x^{(i)},y^{(3)}) - u(x^{(i)},y^{(1)})} \tag{5.2.12}$$

which can be solved for $\theta(x^{(i)})$. Therefore $u(x^{(i)},y)$ is determined by (5.2.11).

Setting $x = x^{(2)}$ in (5.2.10) yields

$$u_x[x^{(2)} \mid \phi(y)] = \frac{u(x^{(2)},y) - u(x^{(1)},y)}{u(x^{(3)},y) - u(x^{(1)},y)} \tag{5.2.13}$$

which may be solved for $\theta(y)$.

Therefore $u(x,y)$ is determined by (5.2.10). Hence the theorem is proved.

* * * * * * *

<u>Example</u>. Suppose the conditions of the last theorem are met with

$$u_x(x \mid \phi) = x^{\phi}$$

and (5.2.14)

$$u_y(y \mid \theta) = y^{\theta}.$$

In order to simplify notation let $k_{ij} \equiv u(x^{(i)}, y^{(j)})$.

From (5.2.11) it follows that

$$u(x^{(i)}, y) = k_{i1} + (k_{i3} - k_{i1})\, y^{\theta(x^{(i)})} \tag{5.2.15}$$

and hence

$$\theta(x^{(i)}) = \frac{\log \left\{ \dfrac{k_{i2} - k_{i1}}{k_{i3} - k_{i1}} \right\}}{\log y^{(2)}} \tag{5.2.16}$$

$$\equiv c_i \;.$$

Thus (5.2.15) can be rewritten as

$$u(x^{(i)}, y) = k_{i1} + (k_{i3} - k_{i1})\, y^{c_i} \tag{5.2.17}$$

Substituting this into (5.2.10) yields

$$\begin{aligned} u(x,y) = {} & k_{11} + (k_{13} - k_{11})\, y^{c_1} \\ & + [k_{13} - k_{11} + (k_{33} - k_{31})y^{c_3} - (k_{13} - k_{11})y^{c_1}]\, x^{\phi(y)} \end{aligned} \tag{5.2.18}$$

and hence

$$\phi(y) = \frac{\log \left\{ \dfrac{k_{21} + (k_{23} - k_{21})y^{c_2} - k_{11} - (k_{13} - k_{11})y^{c_1}}{k_{31} - k_{11} + (k_{33} - k_{31})y^{c_3} - (k_{13} - k_{11})y^{c_1}} \right\}}{\log x^{(2)}} \tag{5.2.19}$$

This can be substituted into (5.2.18) to give $u(x,y)$ for all x and y.

This example involves a lot of messy algebra. The reader may wonder whether this approach to assessing two attribute utility functions is any easier than merely assessing the utilities of a number of different points and fairing a curve through them.

In fact it is since the utilities of only seven points need to be assessed to specify $u(x,y)$ for all x and y. The algebra may be messy but this can be carried out by computer while the analyst concentrates on making sure the utility assessments for the seven points correctly represent the preferences of the decision maker.

If a curve were faired in, in most cases many more than seven points would have to be considered. Thus, the time would usually not be available to make sure the assessment at each point was actually correct.

5.2.3 One Attribute Parametrically Dependent

In this section the case where x is parametrically dependent on y, but there is no restriction on y is considered. It is shown that three (consistently scaled) conditional utility functions over y determine $u(x,y)$ for all x and y.

Theorem 5.2.3. Suppose x is parametrically dependent on y. Then $u(x,y)$ is determined by the parametric form $u_x(x|\theta)$ and three consistently scaled conditional utility functions $u(x^{(i)},y)$, $i = 1,2,3$ (where $x^{(1)} \prec x^{(2)} \prec x^{(3)}$).

Proof. Assume, without loss of generality, that $u_x(x^{(1)}|\theta) = 0$ and $u_x(x^{(3)}|\theta) = 1$).

Since x is parametrically dependent on y it follows that

$$u(x,y) = c_1(y) + c_2(y)\, u_x[x|\theta(y)]. \tag{5.2.20}$$

Therefore

$$u(x,y) = u(x^{(1)},y) + [u(x^{(3)},y) - u(x^{(1)},y)] \times u_x[x|\theta(y)]. \tag{5.2.21}$$

Setting $x = x^{(2)}$ in (5.2.21) yields

$$u_x[x^{(2)} | \theta(y)] = \frac{u(x^{(2)},y) - u(x^{(1)},y)}{u(x^{(3)},y) - u(x^{(1)},y)} \tag{5.2.22}$$

which can be solved for $\theta(y)$. This can be substituted into (5.2.21) to yield $u(x,v)$. Thus the theorem is established.

* * * * * * *

The following theorem shows that one of the conditional utility functions in the last theorem can be replaced by an indifference curve and $u(x,y)$ is still determined.

<u>Theorem 5.2.4</u>. Suppose x is parametrically dependent on y. Then $u(x,y)$ is determined by the parametric form $u_x(x|\theta)$, two consistently scaled conditional utility functions $u(x^{(1)},y)$ and $u(x^{(2)},y)$ where $x^{(1)} \prec x^{(2)}$), and an indifference curve $x = x_I(y)$ with its utility $u[x_I(y),y]$.

<u>Proof</u>. Assume, without loss of generality, that $u_x(x^{(1)}|\theta) = 0$ and $u_x(x^{(2)}|\theta) = 1$.

By exactly the same reasoning as in the last theorem

$$u(x,y) = u(x^{(1)},y) + [u(x^{(2)},y) - u(x^{(1)},y)] \times u_x[x|\theta(y)]. \tag{5.2.23}$$

In order to specify $\theta(y)$ set $x = x_I(y)$ in (5.2.23). This yields

$$u_x[x_I(y)|\theta(y)] = \frac{u[x_I(y),y] - u(x^{(1)},y)}{u(x^{(2)},y) - u(x^{(1)},y)} \tag{5.2.24}$$

which may be solved for $\theta(y)$. This is then substituted into (5.2.23) to give $u(x,y)$ for all x only.

5.3 Parametric Dependence for N-Attribute Utility Functions

In this section some of the results for two attribute utility functions are extended to the n-attribute case. Theorem 5.3.1 considers cases where each attribute is parametrically dependent on its complement while theorem 5.3.2 looks at cases where some attributes are utility independent of their complements and others are parametrically dependent on their complements.

It is shown that when all the attributes are parametrically dependent then $u(x_1, x_2, \ldots, x_n)$ is specified by the n functional forms $u_i(x_i|\theta_i)$, $i = 1, 2, \ldots, n$ and the utilities of $3^n - 2$ points. When m of the attribute are utility independent then the utilities of $2^m \cdot 3^{n-m} - 2$ points are needed. Before proceeding some useful notation is established. Let

$$(\underline{x}_{k-1}; \underline{x}^{(\underline{i})}_{k+1}) \equiv (x_1, x_2, \ldots, x_{k-1}, x_{k+1}^{(i_{k+1})}, x_{k+2}^{(i_{k+2})}, \ldots, x_n^{(i_n)})$$

and

$$(\underline{x}_{k-1}; x_k; \underline{x}^{(\underline{i})}_{k+1}) \equiv (x_1, x_2, \ldots, x_k, x_{k+1}^{(i_{k+1})}, x_{k+2}^{(i_{k+2})}, \ldots, x_n^{(i_n)}).$$

Theorem 5.3.1. Suppose for all k that x_k is parametrically dependent on $x_{\bar{k}}$ for $x_{\bar{k}} = (\underline{x}_{k-1}; x^{(\underline{i})}_{k+1})$, where $i_{k+1}, i_{k+2}, \ldots, i_n = 1, 2, 3$. Then $u(x_1, x_2, \ldots, x_n)$ is determined by the n parametric forms $u_k(x_k|\theta_k)$, $k = 1, 2, \ldots, n$ and the utilities of any $3^n - 2$ of the points $(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_n^{(i_n)})$, $i_k = 1, 2, 3$ for $k = 1, 2, \ldots, n$ (where $x_k^{(1)} \prec x_k^{(2)} \prec x_k^{(3)}$ for all k).

Proof. Assume, without loss of generality, that $u_k(x_k^{(1)}|\theta_k) = 0$ and $u_k(x_k^{(3)}|\theta_k) = 1$ for all k.

Since x_k is parametrically dependent on $x_{\bar{k}}$ for $x_{\bar{k}} = (\underline{x}_{k-1}; x^{(i)}_{k+1})$

then

$$u(\underline{x}_{k-1}; x_k; \underline{x}^{(i)}_{k+1}) = C_{1k}(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1}) \tag{4.3.1}$$

$$+ C_{2k}(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1})\ u_k[x_k \,|\, \theta(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1})].$$

From this it follows that

$$u(\underline{x}_{k-1}; x_k; \underline{x}^{(i)}_{k+1}) = u(\underline{x}_{k-1}; x^{(1)}_k; \underline{x}^{(i)}_{k+1}) \tag{4.3.2}$$

$$+[u(\underline{x}_{k-1}; x^{(3)}_k; \underline{x}^{(i)}_{k+1}) - u(\underline{x}_{k-1}; x^{(1)}_k; \underline{x}^{(i)}_{k+1})]\ u_k[x_k \,|\, \theta(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1})].$$

Consider $k = 1$. Then, from (4.3.2), $u(x_1, x_2^{(i_2)}, x_3^{(i_3)}, \ldots, x_n^{(i_n)})$ would be determined if $\theta(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1})$ were known. Set $x_k = x_k^{(2)}$ in (4.3.2). Then

$$u_k[x_k^{(2)} \,|\, \theta(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1})] = \frac{u(\underline{x}_{k-1}; x^{(2)}_k; \underline{x}^{(i)}_{k+1}) - u(\underline{x}_{k-1}; x^{(1)}_k; \underline{x}^{(i)}_{k+1})}{u(\underline{x}_{k-1}; x^{(3)}_k; \underline{x}^{(i)}_{k+1}) - u(\underline{x}_{k-1}; x^{(1)}_k; \underline{x}^{(i)}_{k+1})}. \tag{4.3.3}$$

This may be solved for $\theta(\underline{x}_{k-1}; \underline{x}^{(i)}_{k+1})$ if $k = 1$.

Having solved (4.3.2) to yield $u(\underline{x}_{k-1}; x_k; \underline{x}^{(i)}_{k+1})$ for $k = 1$, it may now be solved iteratively for $k = 2, 3, \ldots, n$. Thus $u(x)$ is determined.

* * * * * * *

<u>Theorem 5.3.2.</u> Suppose that for $k \leq m$ x_k is parametrically dependent on $x_{\bar{k}}$ for $x_{\bar{k}} = (\underline{x}_{k-1}; x^{(i)}_{k+1})$, where $i_{k+1}, i_{k+2}, \ldots, i_n = 1, 2, 3$, and suppose further that for $k > m$ x_k is utility independent of $x_{\bar{k}}$. Then $u(x_1, x_2, \ldots, x_n)$ is determined by the following:

i) the parametric forms $u_k(x_k|\theta)$, $k = 1,2,\ldots,m$,

ii) conditional utility functions $u_k(x_k)$, $k = m+1$, $m+2,\ldots,n$, and

iii) the utilities of any $3^m \cdot 2^{n-m}-2$ of the points $(x_1^{(i_1)}, x_2^{(i_2)},\ldots,x_n^{(i_n)})$ where $i_k = 1,2,3$ for $k = 1,2,\ldots,m$ and $i_k = 1,2$ for $k = m+1$, $m+2,\ldots,n$

(where $x_i^{(1)} \prec x_i^{(2)} \prec x_i^{(3)}$ for all i).

<u>Proof</u>. Assume without loss of generality that $u_k(x_k^{(1)}|\theta_k) = 0$ and $u_k(x_k^{(3)}|\theta_k) = 1$ for all $k \leq m$ and that $u_k(x_k^{(1)}) = 0$ and $u_k(x_k^{(2)}) = 1$ for all $k > m$.

Then the proof of theorem 5.3.1 establishes that $u(\underline{x}_{m-1};x_m;\underline{x}_{m+1}^{(\underline{i})})$ is known.

For $k > m$

$$u(\underline{x}) = C_{1k}(x_{\bar{k}}) + C_{2k}(x_{\bar{k}})\, u(x_k). \tag{5.3.4}$$

Therefore

$$\begin{aligned} u(\underline{x}_{k-1};x_k;\underline{x}_{k+1}^{(\underline{i})}) &= u(\underline{x}_{k-1};x_k^{(1)};\underline{x}_{k+1}^{(\underline{i})}) \\ &+ [\,u(\underline{x}_{k-1};x_k^{(2)};\underline{x}_{k+1}^{(\underline{i})}) - u(\underline{x}_{k-1};x_k^{(1)};\underline{x}_{k+1}^{(\underline{i})})]\, u_k(x_k) \end{aligned} \tag{5.3.5}$$

Since $u(\underline{x}_{m-1};x_m;\underline{x}_{m+1}^{(\underline{i})})$ is known, (5.3.5)

can be solved for $u(\underline{x}_m;x_{m+1};\underline{x}_{m+2}^{(\underline{i})})$.

This procedure can be continued iteratively until $k = n$ at which point $u(\underline{x})$ is determined.

The theorems in this section and the last have shown how multiattribute utility functions may be approximated using the utilities of a relatively small number of points. As was pointed out earlier in the chapter, the need for only a small number of utilities means that care can be taken in the assessment of these to make sure they accurately reflect the person's preferences.

This approach to utility assessment is particularly valuable for a decision maker who wishes to incorporate the preferences of a large number of people into his analysis, since, if he uses it, he can approximately assess their utility functions fairly rapidly.

However, if the utilities of the various individuals are <u>interdependent</u> then it is necessary to account for this interdependence in addition to assessing the utility functions over various outcomes $\underline{x}$. This problem is considered in the next section.

5.4 Interdependent Utility Functions

As was noted at the beginning of this chapter, the preferences of an individual in the group of people whose views are important to a decision maker will often depend on the preferences of the decision maker and the other people in the group. That is, $u_i = u_i(\underline{x};U;u_{\bar{i}})$. This leads to the set of equations

$$
\begin{aligned}
U &= U(\underline{x};\underline{u}) \\
u_1 &= u_1(\underline{x};U;u_{\bar{1}}) \\
u_2 &= u_2(\underline{x};U;u_{\bar{2}}) \\
&\vdots \\
u_n &= u_n(\underline{x};U;u_{\bar{n}})
\end{aligned}
\qquad (5.4.1)
$$

In section 4.1 it was noted that often $\underline{x}$ and $\underline{u}$ would be mutually utility independent for the decision maker so that

$$U = K_1 U_x(\underline{x}) + K_2 U_u(\underline{u}) + K_3 U_x(\underline{x}) U_u(\underline{u}) \qquad (5.4.2)$$

where $U_x(\underline{x})$ and $U_u(\underline{u})$ are conditional utility functions and K_1, K_2 and K_3 are constants.

By the same arguments $\underline{x}$ and $\{U; u_{\bar{i}}\}$ would often be mutually utility independent for each of the individuals in the group. In this case

$$u_i = K_{i1} u_{ix}(\underline{x}) + K_{i2} u_{iu}(U; u_{\bar{i}}) + K_{i3} u_{ix}(\underline{x}) u_{iu}(U; u_{\bar{i}}) \qquad (5.4.3)$$

for $i = 1, 2, \ldots, n$, where $u_{ix}(\underline{x})$ and $u_{iu}(U; u_{\bar{i}})$ are conditional utility functions, and K_{i1}, K_{i2} and K_{i3} are constants.

In section 4.3 it was shown that often for the decision maker $u_1, u_2, \ldots, u_n$ would be order-one mutually utility independent with conditional utility functions linear in the u_i's, and that they would be order-two mutually preferentially independent. In this case either

$$U_u(\underline{u}) = K^{-1}\{\prod_{i=1}^{n}(Kk_i u_i + 1) - 1\} \qquad (5.4.4)$$

or

$$U_u(\underline{u}) = \sum_{i=1}^{n} k_i u_i \qquad (5.4.5)$$

where $K, k_1, k_2, \ldots, k_n$ are constants.

By the same arguments $U, u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$ would often be order-one mutually utility independent for the group members with conditional

utility functions linear in the u_i's, and also order-two mutually preferentially independent. In this case either

$$u_{iu}(U;u_{\bar{i}}) = C_i^{-1}\{(C_i c_{io} U+1) \prod_{\substack{i=1\\ k\neq i}}^{n} (C_i c_{ik} u_i + 1) - 1\} \tag{5.4.6}$$

or

$$u_{iu}(U;u_{\bar{i}}) = \sum_{\substack{k=1\\ k\neq i}}^{n} c_{ik} u_k \tag{5.4.7}$$

for $i = 1, 2, \ldots, n$, where $C_i, c_{io}, c_{i1}, \ldots, c_{in}$ are constants.

The constants in equations (5.4.3), (5.4.6) and (5.4.7) could be evaluated in the same way as those in equations (5.4.2), (5.4.4) and (5.4.5). (This problem was considered in section 4.5.)

The conditional utility functions $u_{ix}(\underline{x})$, $i = 1, 2, \ldots, n$ could be approximately evaluated using the methods discussed in sections 5.2 and 5.3.

In order to find the decision maker's utility U for a particular outcome $\underline{x}$ it would be necessary to simultaneously solve the system of equations (5.4.2) - (5.4.7). In general this could not be done analytically. However, numerical methods could probably be worked out to solve the problem.

Considering one special case will point out how unexpected results may occur when the preferences of others are incorporated into a utility function. Suppose there is one person whose views are of interest to the decision maker and one attribute x which describes the outcomes. Then, if (5.4.2) and (5.4.3) are accepted,

$$U = K_1U_x(x)+K_2u+K_3uU_x(x) \tag{5.4.8}$$

and

$$u = k_1u_x(x)+k_2U+k_3Uux(x) \tag{5.4.9}$$

where $U_x(x)$ and $u_x(x)$ are conditional utility functions and K_1, K_2, K_3, k_1, k_2 and k_3 are constants.

Suppose that all the constants are 1 except k_2 and k_3 and that these are zero, and also that $U_x(x) = u_x(x) = x$. Then

$$U = x+u+xu \tag{5.4.10}$$

and

$$u = x. \tag{5.4.11}$$

Notice that both of the individuals are risk neutral toward lotteries over x when the preferences of the other person are held fixed.

If (5.4.10) and (5.4.11) are solved to yield U as a function of x then

$$U = 2x + x^2. \tag{5.4.12}$$

For positive values of x this utility function is risk prone toward lotteries over x. Thus, even though both individuals are risk neutral in their direct preferences for outcomes, the fact that the decision maker takes into account the preferences of the other person makes his total preferences for outcomes risk prone.

Chapter V Footnotes

1. Grochow[9] has noted how time consuming this is.

2. See Meyer and Pratt[23] and Spetzler[33].

3. See Pratt[26].

4. See Howard[14], p. 513.

5. See Kaufman[15].

6. See Spetzler[33].

7. See Keeney[18].

8. See Keeney[16].

9. See Keeney[18].

Chapter VI

ACCOUNTING FOR UNCERTAINTY ABOUT THE PREFERENCES OF GROUP MEMBERS

As noted before, it is often impossible for a decision maker to be certain of the preferences of all individuals of interest to him. The resources may not be available to assess everyone's preferences. Some individuals may deliberately misrepresent their views. People may give incorrect preferences because they have not thought hard enough about what their true preferences are.

In this chapter methods are developed for dealing with uncertainty about preferences. A statistical decision theory approach is taken.[1] That is, the decision maker's state of knowledge is summarized in a subjective probability distribution and Bayes' theorem is used to update this in light of new information.

In this chapter it will be assumed that the results of sections 4.1 and 4.3.2 hold so that

$$U(\underline{x};u) = K_1 U_x(\underline{x}) + K_2 U_u(\underline{u}) + K_3 U_x(\underline{x}) U_u(u)$$

and (6.0.1)

$$U_u(\underline{u}) = K^{-1}\{\prod_{i=1}^{n} (Kk_i u_i + 1) - 1\}. \quad (6.0.2)$$

The chapter divides into two parts. Part A considers situations where the decision maker has no direct preferences for outcomes. That is, $U = U(\underline{u})$. (Of course, U depends indirectly on the outcomes $\underline{x}$ since the u_i's will depend on $\underline{x}$.) This situation is of interest because it is a case that is important in

applications, and also because many of the results obtained for it can be generalized to the case where U depends also on $\underline{x}$. This more general case is discussed in Part B.

Within Part A two different situations are studied. In section 6.1 it is shown that the u_i's might be probabilistically independent in some cases. Results are derived for this situation. In section 6.2, u_i's that are probabilistically dependent are considered.

Part A

Decision Makers with no Direct Preferences for Outcomes

A decision maker might have a utility function $U(\underline{u})$ in two cases. First, when he is serving purely as a servant of the group and is relaying its preferences without accounting for his own. This occurs in Application A in the next chapter. There the decision maker is conducting a group discussion and then recording the preferences of the group to be passed on to a government body. He does not let his own preferences for outcomes influence the preference measure at all.

Another case where the utility function $U(\underline{u})$ might be used is when decision analytic approaches are only being used to analyze the preferences $u_1, u_2, \ldots, u_n$, but some other method of analysis is being used for the rest of the study. This is the case in Application B in the next chapter. There the utility theory approach is taken to finding the preferences of computer time-share system users for different system characteristics. However, decision

analysis is not necessarily used to incorporate these preferences into a complete system design or evaluation scheme.

6.1 Probabilistically Independent u_i's

Suppose $u_1, u_2, \ldots, u_n$ are utility functions representing the preferences of distinct groups of people. If these groups have fairly well defined viewpoints which differ from each other, then a decision maker may feel that information about the preferences of one or more of the groups will not alter his subjective probability distribution for the preferences of the other groups. In other words, $u_1, u_2, \ldots, u_n$ will be mutually probabilistically independent.

In this case many simplifications occur in the mathematics involved in considering uncertainty. Since it seems that some practical application situations would be of this type, it will be considered in some detail.

6.1.1 Situations Where the $\underline{x}_k$'s are Certain

If the outcome $\underline{x}_k$ that results from each possible action a_k, $k = 1,2,\ldots,r$ is known for sure then the utility $U(a_k)$ of that action is

$$U(a_k) = E_{u|x}[U|\underline{x}_k] \tag{6.1.1}$$

for $k = 1,2,\ldots,r$, where $E_{u|x}[\cdot|\cdot]$ is the conditional expected value of U given $\underline{x}_k$.

When (6.0.2) holds then this becomes

$$U(a_k) = E_{u|x}[K^{-1}\{\prod_{i=1}^{n}(Kk_iu_i+1)-1\}|\underline{x}_k] \tag{6.1.2}$$

If, in addition, the u_i's are mutually utility independent, then

$$U(a_k) = K^{-1}\{\prod_{i=1}^{n} (Kk_i E_{u|x}[u_i|\underline{x}_k]+1)-1\} . \qquad (6.1.3)$$

Notice that $U(a_k)$ depends only on the conditional expected values of the u_i's. Although it may be necessary to assess the conditional probability distributions over each u_i to obtain these expected values, it is not necessary to assess a joint probability distribution over the u_i's. This follows, of course, from the fact that the u_i's are mutually probabilistically independent.

6.1.1.1 Sample Information With no Bias

In this section $E_{u|x}[\cdot|\cdot]$ will be abbreviated as $E[\cdot|\cdot]$ for notational simplicity. Also the entity having utility function u_j will be called a "group" rather than an individual since as pointed out before, that is the case when probabilistic independence of the u_i's might hold.

If the true, unbiased utilities $u_j(a_1), u_j(a_2), \ldots, u_j(a_r)$ of the $j^{\underline{th}}$ group for all the possible actions are obtained, then the updated $U(a_k)$ utilizing this sample data S is

$$U(a_k|S) = K^{-1}\{[Kk_j u_j(a_k)+1] \prod_{\substack{i=1 \\ i\neq j}}^{n} (Kk_i E[u_i|x_k]+1)-1\} . \qquad (6.1.4)$$

The manner in which the sample data can influence things may be illustrated by considering the case where $n = 2$ and $r = 2$. The (6.1.4) reduces to

$$U(a_k|S) = K^{-1} \{[Kk_1 u_1(a_k)+1] [Kk_2 E(u_2|\underline{x}_k)+1]-1\} \qquad (6.1.5)$$

for $k = 1,2$.

It follows from this that $a_1 \left\{\begin{matrix} \succ \\ \sim \\ \prec \end{matrix}\right\} a_2$ if and only if

$$K^{-1}\{[Kk_1u_1(a_1)+1][Kk_2E(u_2|\underline{x}_1)+1]\}$$

$$\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} K^{-1}\{[Kk_1u_1(a_2)+1][Kk_2E(u_2|\underline{x}_2)+1]-1\}. \tag{6.1.6}$$

If $K > 0$ then this reduces to

$$[Kk_1u_1(a_1)+1][Kk_2E(u_2|\underline{x}_1)+1]$$

$$\left\{\begin{matrix} > \\ = \\ < \end{matrix}\right\} [Kk_1u_1(a_2)+1][Kk_2E(u_2|\underline{x}_2)+1]. \tag{6.1.7}$$

(For $K < 0$ the directions of the inequalities are reversed.)

The values of $u_1(a_1)$ and $u_1(a_2)$ for which a_1 or a_2 will be the preferred action of the decision maker are shown in figure 6.1. It is interesting that the region where a_1 is preferred is divided from the region where a_2 is preferred by a straight line. Notice that the larger the margin by which $E(u_2|\underline{x}_2)$ exceeds $E(u_2|\underline{x}_1)$ then the larger the margin must be between $u_1(a_2)$ and $u_1(a_1)$ before a_2 becomes the preferred action. That is, if the group whose views are uncertain is "expected" to favor a_1 by a large margin then the group whose views are measured must favor a_2 by a large margin to overcome this expectation.

Also, it is clear that the greater the importance of the first group's views to the decision maker (i.e., the greater value of k_1) then the more nearly the decision maker will have the same preference regions as that group.

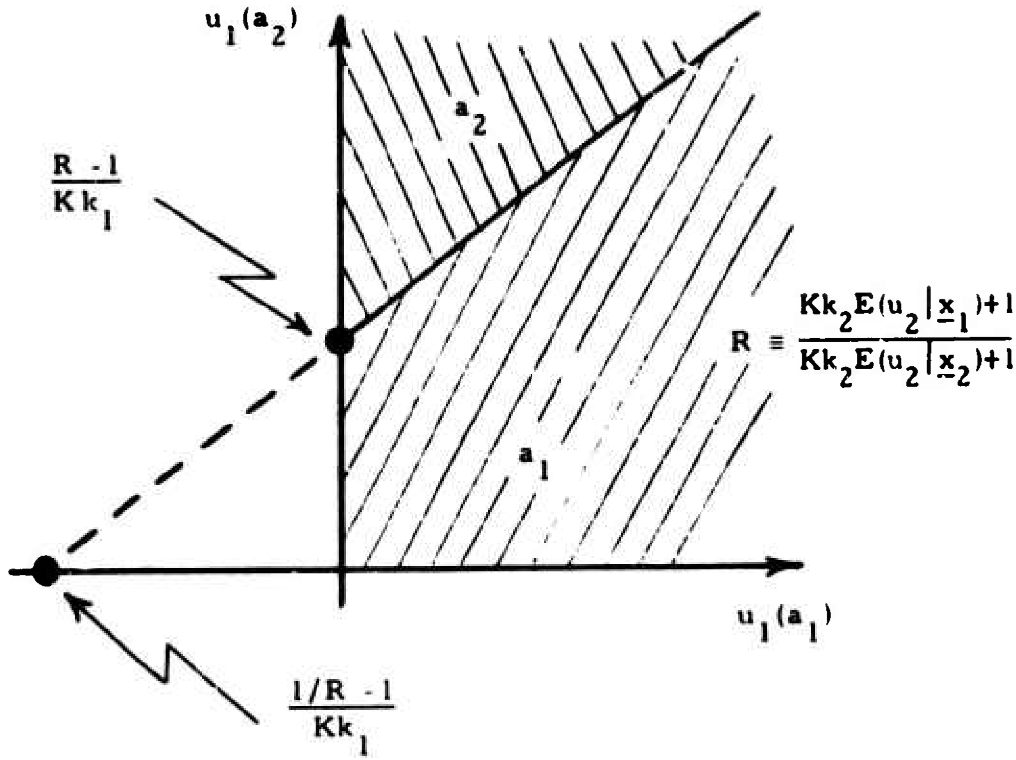


Figure 6.1. Preference Regions Using Sample Data

Returning now to the case where r and n are general as in equation (6.1.4) then it is e. ιy to generalize to the case where the preferences of more than one group are sampled. Thus if a sample S consisting of the preferences of the $j^{\underline{th}}$ and $\ell^{\underline{th}}$ groups is obtained then

$$U(a_k|S) = K^{-1}\{[Kk_j u_j(a_k)+1][Kk_\ell u_\ell(a_k)+1] \times \prod_{\substack{i=1\\ i\neq j,\ell}}^{n} (Kk_i E[u_i|\underline{x}_k]+1)-1\}. \tag{6.1.8}$$

6.1.1.2 Sample Information With Bias

Suppose that any group's measured utility function will not represent the true of preferences of that group either because of a deliberate atten.pt to conceal preferences or because the group has not given careful enough thought to the utility assessment. Suppose that $u_{im}(a_k)$ is the measured utility of action a_k to the $i^{\underline{th}}$ group. Then the bias of this measurement is defined as[2]

$$b_i(a_k) \equiv u_{im}(a_k)-u_i(a_k)$$

where $u_i(a_k)$ is the true utility of a_k tc the $i^{\underline{th}}$ group.

If $b_i(a_k)$ were known for certain then the correction for bias would be easy. Equation (6.1.4), giving the utility of a_k when the preferences of one group have been measured, would become

$$U(a_k|S) = K^{-1}\{[Kk_j\{u_{jm}(a_k)-b_j(a_k)\}+1] \times \prod_{\substack{i=1\\ i\neq j}}^{n} (Kk_i E[u_i|\underline{x}_k]+1)-1|a_k\} \tag{6.1.9}$$

However, if the bias is uncertain then this uncertainty must be accounted for. In this case

$$U(a_k|S) = K^{-1} E\{Kk_j\{u_{jm}(a_k) - b_j(a_k)\} + 1]$$
$$\times \prod_{\substack{i=1 \\ i\neq j}}^{r} (Kk_i u_i + 1) - 1 | a_k\} . \tag{6.1.10}$$

When $b_j(a_k)$ is probabilistically independent of $u_i, i=1,2,\ldots j-1, j+1,\ldots,n$ and, as before, the u_i's are mutually probabilistically independent, then (6.1.10) reduces to

$$U(a_k|S) = K^{-1}\{[Kk_j\{u_{jm}(a_k) - E(b_j|a_k)\}+1]$$
$$\times \prod_{\substack{i=1 \\ i\neq j}}^{n} (Kk_i E[u_i|\underline{x}_k]+1) - 1\} . \tag{6.1.11}$$

Notice that $b_j(a_k)$ does not have to be probabilistically independent of $u_{jm}(a_k)$ for this result to hold.

The effect of bias may be illustrated by considering the case where there are two groups and two possible actions. If the preferences of the first group are measured then (6.1.11) reduces to

$$U(a_k|S) = K^{-1}\{[Kk_1\{u_{1m}(a_k) - E(b_1|a_k)\}+1]$$
$$\times (Kk_2 E[u_2|\underline{x}_k]+1) - 1\} . \tag{6.1.12}$$

It follows from this that $a_1 \left\{ \begin{matrix} \succ \\ \sim \\ \prec \end{matrix} \right\} a_2$ if and only if

$$K^{-1}\{[Kk_1\{u_{1m}(a_1)-E(b_1|a_1)\}+1](Kk_2E[u_2|\underline{x}_1]+1)-1\}$$

$$\begin{Bmatrix}>\\=\\<\end{Bmatrix} K^{-1}\{[Kk_1\{u_{1m}(a_2)-E(b_1|a_2)\}+1](Kk_2E[u_2|\underline{x}_2]+1)-1\}. \tag{6.1.13}$$

If $K > 0$ this reduces to

$$[Kk_1\{u_{1m}(a_1)-E(b_1|a_1)\}+1](Kk_2E[u_2|\underline{x}_1]+1)$$

$$\begin{matrix}>\\=\\<\end{matrix} \quad [Kk_1\{u_{1m}(a_2)-E(b_1|a_2)\}+1](Kk_2E[u_2|\underline{x}_2]+1) \tag{6.1.14}$$

(If $K < 0$ then the directions of the inequalities are reversed.) The equation which corresponds to this for the unbiased case is (6.1.7):

$$[Kk_1u_1(a_1)+1]\,[Kk_2E(u_2|\underline{x}_1)+1]$$

$$\begin{Bmatrix}>\\=\\<\end{Bmatrix} [Kk_1u_1(a_2)+1][Kk_2E(u_2|\underline{x}_2)+1]. \tag{6.1.15}$$

Comparing this with (6.1.14) shows that the effect of uncertain bais is roughly what would be expected. Suppose, for example, that $E(b_1|a_2) = 0$ and $E(b_1|a_1) > 0$. Then a larger value $u_{1m}(a_1)$ must be obtained for a_1 to be preferred then was necessary without bias. In other words, if it is "expected" that a utility for a_1 that is higher than the true value will be measured then this is compensated for by requiring a larger value to be measured before a_1 becomes the preferred action.

6.1.2 Situations Where the $\underline{x}_k$'s are Uncertain

If the $\underline{x}$ that results from any a_k is uncertain then

$$U(a_k) = E_{x|a}\{E_{u|x}[U(\underline{u})|x]|a_k\} . \tag{6.1.16}$$

When (6.0.2) holds and the u_i's are probabilistically independent then

$$U(a_k) = E_{x|a}\{K^{-1}[\prod_{i=1}^{n}(Kk_i E[u_i|\underline{x}]+1)-1]|a_k\}. \tag{6.1.17}$$

Thus $U(a_k)$ depends on the conditional expected values $E[u_i|x]$ and, in addition, on the probability distribution of $\underline{x}$. Note, however, that $E[u_i|x]$'s must be assessed for all possible x's. This may be a large number since the $\underline{x}$ resulting from each a_k is uncertain. Thus the assessment problem may be difficult.

In some cases it may be reasonable to assume a special form for $E[u_i|\underline{x}]$. Thus, for example, it might be true in some situations that

$$E[u_i|\underline{x}] = \underline{a}_i(\underline{x}^T - \underline{x}_{io}^T) \tag{6.1.18}$$

where $\underline{a}_i$ and $\underline{x}_{io}$ are vector constants and the superscript T indicates a transpose. Then it would only be necessary to assess $\underline{a}_i$ and $\underline{x}_{io}$ to specify $E[u_i|\underline{x}]$.

Sample information, both biased and unbiased, can be treated in a manner similar to what was done when the $\underline{x}$ resulting from any action was certain. Thus, if an unbiased measurement of $u_j(\underline{x})$ is available then this sample data S may be used to yield

$$\begin{aligned} U(a_k|S) = E_{x|a}\{K^{-1}[(Kk_j u_j(\underline{x})+1) \\ \times \prod_{\substack{i=1 \\ i\neq j}}^{n}(Kk_j E[u_i|\underline{x}]+1)-1]|a_k\}. \end{aligned} \tag{6.1.19}$$

Similarly, if a biased measurement $u_{jm}(\underline{x})$ is made and the bias is probabilistically independent of the other utilities $u_i(\underline{x})$, $i \neq j$, then

$$U(a_k|S) = E_{x|a}\{K^{-1}[(Kk_j\{u_j(\underline{x})-E(b_j|\underline{x})\}+1) \times \prod_{\substack{i=1 \\ i\neq j}}^{n} (Kk_iE[u_i|\underline{x}]+1)-1]|a_k\}. \tag{6.1.20}$$

Equations (6.1.19) and (6.1.20) are direct generalizations of (6.1.8) and (6.1.11) to the case where there is uncertainty in $\underline{x}$.

6.2 Probabilistically Dependent u_i's

The situation where $u_1, u_2, \ldots, u_n$ are mutually probabilistically independent was studied in the last section. However, often the u_i's would be probabilistically dependent. That is, information about the values of one or more of the u_i's would change the decision maker's subjective probability distribution for the others.

In section 6.2.1 some general results for this case are derived, and in section 6.2.2 some useful special cases are examined.

6.2.1 General Results

If the $\underline{x}$ that results from any action a_k is known for sure then, as shown in equation (6.1.2)

$$U(a_k) = E_{u|x}\{K^{-1}[\prod_{i=1}^{n}(Kk_iu_i+1)-1]|\underline{x}_k\}. \tag{6.2.1}$$

The meaning of this may be illustrated by considering the case where $n = 2$. Then (6.2.1) reduces to

$$U(a_k) = Kk_1k_2E(u_1u_2|\underline{x}_k)+k_1E(u_1|\underline{x}_k)+k_2E(u_2|x_k). \qquad (6.2.2)$$

Thus it is necessary to assess the expected values of u_1 and u_2 conditional on $\underline{x}_k$ as well as the conditional cross-correlation between them. In most cases the only way the cross-correlation could be obtained would be to assess the joint probability distribution for u_1 and u_2 conditional on $\underline{x}_k$. Since this distribution is needed for each possible $\underline{x}_k$ the assessment problem becomes very difficult if there are very many possible actions $a_1, a_2, \ldots, a_r$ being considered.

If there are more than two u_i's then even more data is needed to specify $U(a_k)$. It is clear from (6.2.2) that all of the conditional cross-moments between the u_i's up to $n^{\underline{th}}$ order are needed. In most situations these could be obtained only by assessing the joint probability distribution over $u_1, u_2, \ldots, u_n$. If n is very large this will be a formidable task.

If there is uncertainty about the $\underline{x}$ that results from each a_k as well as about the utilities $u_1, u_2, \ldots, u_n$ then

$$U(a_k) = E_{x|a}[E_{u|x}\{K^{-1}[\prod_{i=1}^{n}(Kk_iu_i+1)-1]|\underline{x}\}|a_k\}. \qquad (6.2.3)$$

This requires the assessment of the cross-moments between the u_i's for all possible values of $\underline{x}$. Since $\underline{x}$ is uncertain this may be a very large number. Hence the assessment would be very difficult.

In the next section some special structured situations where $U(a_k)$ can be determined relatively easily are studied.

6.2.2 Some Special Structured Situations

Suppose that $U(u)$ is symmetric with respect to the attributes $u_1, u_2, \ldots, u_n$. (This case was studied in sections 4.2 and 4.4.) Then (6.0.2) reduces to

$$U(\underline{u}) = K^{-1}\{\prod_{i=1}^{n} (Kku_i+1)-1\} \tag{6.2.4}$$

where K and k are constants.

This situation is studied in the next two sections.

6.2.2.1 Situations Where the $\underline{x}_k$'s are Certain

Suppose the number of people that had each possible utility for the $\underline{x}_k$ that results from any action a_k were known. If $p_{u|x}(u_j|\underline{x}_k)$ is the fraction of the total number of people that has utility u_j for outcome $\underline{x}_k$, then from (6.2.4) it follows that

$$U(a_k) = K^{-1}\{\prod_{\text{all } j} [Kku_j+1]^{np_{u|x}(u_j|\underline{x}_k)} -1\}. \tag{6.2.5}$$

This may be rewritten as

$$U(a_k) = K^{-1}\{\exp[nE_{u|x}\{\log_e(Kk_u+1)|\underline{x}_k\}]-1\} \tag{6.2.6}$$

where

$$E_{u|x}\{\log_e(Kk_u+1)|\underline{x}_k\}$$

$$= \sum_{\text{all } j} p_{u|x}(u_j|\underline{x}_k) \log_e(Kku_j+1). \tag{6.2.7}$$

Equation (6.2.6) is not very useful as it stands since, under conditions of uncertainty, $p_{u|x}(\cdot\,|\,\cdot)$ is not known. In some cases, however, the decision maker may be willing to assume that $p_{u|x}(\cdot\,|\,\cdot)$ would be known if the value of some uncertain parameter θ were known. That is, $p_{u|x,\theta}(\cdot\,|\underline{x}_k,\theta)$ is assumed to be known. For example, the decision maker might assume that u is distributed normally with known variance $\sigma_{\underline{x}}^2$ and uncertain mean θ. In that case $p_{u|x,\theta}(\cdot\,|\underline{x},\theta)$ would be a density function given by

$$p_{u|x,\theta}(u|\underline{x},\theta) = \frac{\exp\{-[u-\theta]^2/2\sigma_{\underline{x}}^2\}}{\sigma_{\underline{x}}\sqrt{2\pi}} . \tag{6.2.8}$$

If $p_{u|\underline{x},\theta}(\cdot\,|\underline{x}_k,\theta)$ is known then, from (6.2.7) it follows that

$$U(a_k) = K^{-1}E_{\theta|x}\{\exp[nE_{u|x,\theta}\{\log_e(Kku+1)|\underline{x}_k,\theta\}]-1|\underline{x}_k\} . \tag{6.2.9}$$

This can be assessed if $p_{u|x,\theta}(\cdot\,|\underline{x}_k,\theta)$ and $p_{\theta|x}(\cdot\,|\underline{x}_k)$ are known for all k. If the number of possible actions (and hence the number of possible x_k's) is relatively small then it should be feasible to determine these.

If sample data about the preferences of some individuals is available then it is relatively easy to update (6.2.9) to account for this sample data. Suppose, for simplicity of exposition, that the decision maker wishes to obtain information about the preferences of group members for only one a_k. (It would probably be necessary to assess any individual's utilities for all a_i's in order to obtain his utility for a_k. However, assume for the moment that only the information about a_k is used.)

One of the most common sampling procedures is random sampling with replacement.[3] When this procedure is used each member of the population is equally likely to be sampled each time an element is drawn from the population. Using this procedure there is some probability that the same person will be selected more than once. If the number of people in the group is large this is not very likely to occur. If it does, then, of course, it is not necessary to obtain the person's preferences again since they will already have been determined.

Suppose a sample S of r utilities $u_1(\underline{x}_k)$, $u_2(\underline{x}_k), \ldots, u_r(\underline{x}_k)$ is obtained using random sampling with replacement. Then

$$\underline{P}(S|\underline{x}_k) = \prod_{i=1}^{r} p_{u|x}[u_i(x_k)|\underline{x}_k] \tag{6.2.10}$$

where $\underline{P}(S|\underline{x}_k)$ is the probability of obtaining the sample observed. Then from Bayes' theorem it follows that

$$p_{\theta|x,S}(\theta|\underline{x}_k, S) = \frac{\prod_{i=1}^{r} p_{u|x,\theta}[u_i(\underline{x}_k)|\underline{x}_k, \theta]p_{\theta|x}(\theta|\underline{x}_k)}{\prod_{i=1}^{r} \int_{\theta} p_{u|x,\theta}[u_i(\underline{x}_k)|\underline{x}_k, \tilde{\theta}]p_{\theta|x}(\tilde{\theta}|x_k)d\tilde{\theta}} . \tag{6.2.11}$$

Therefore, the decision maker's utility for a_k given the sample S is

$$U(a_k|S) = K^{-1}E_{\theta|x,S}\{\exp[nE_{u|x,\theta}\{\log_e(Kku+1)|\underline{x}_k, \theta\}]-1|\underline{x}_k, S\}. \tag{6.2.12}$$

Equations (6.2.11) and (6.2.12) are somewhat complicated algebraically, however, there are no conceptual difficulties with them. The complicated

arithmetic needed to evaluate them can be carried out fairly easily by computer.

Equations (6.2.11) and (6.2.12) were derived for the case where sample information for only one a_k was used. The situation is more complicated when several a_k's are of interest. Usually the utilities for different a_k's for a single individual would be probabilistically interdependent. Thus any calculations concerning the results of sampling, like those shown in equations (6.2.11) and (6.2.12), would involve the joint probability distribution for the utilities of any sampled individual over all values of $\underline{x}_k$ that are of interest. Usually it would be very difficult to assess this joint distribution.

One approach that might be used to avoid this problem is to independently sample people for each $\underline{x}_k$ of interest. If this is done then the interdependence of utilities for different $\underline{x}_k$'s would be eliminated and equations (6.2.11) and (6.2.12) could be used for each value of $\underline{x}_k$. Unfortunately, this approach would often involve sampling the preferences of many more people than would be necessary if the utilities of each person for all of the $\underline{x}_k$'s were used.

6.2.2.2 Situations Where the $\underline{x}_k$'s are Uncertain

Suppose that the possible outcomes of the decision making process can be adequately described by a scalar x. Then in some cases the decision maker might feel that each $u_i(x)$ could be adequately represented as being a member of a family of functions with a free parameter θ:

$$u_i(x) = u(x|\theta_i) \tag{6.2.13}$$

For example, it might be reasonable to assume that all the u_i's were exponential

$$u_i(x) = -e^{-\theta_i x} \tag{6.2.14}$$

with only the value of θ_i differing from individual to individual.

If the number of people with each value of θ were known then, from (6.2.4) it would follow that

$$U(x) = K^{-1}\{\prod_{\text{all } i} [Kku(x|\theta_i)+1]^{np_\theta(\theta_i)} - 1\} \tag{6.2.15}$$

where $p_\theta(\theta_i)$ is the fraction of the group with parameter value θ_i. This can be written

$$U(x) = K^{-1}\{\exp[nE_\theta\{\log_e[Kku(x|\theta)+1]\}]-1\} \tag{6.2.16}$$

where

$$E_\theta\{\log_e[Kku(x|\theta)+1]\} = \sum_{\text{all } i} p_\theta(\theta_i)\log_e[Kku(x|\theta_i)+1] \tag{6.2.17}$$

Under uncertainty $p_\theta(\theta_i)$ would not be known and hence (6.2.16) would not be of use to the decision maker. However, in some cases he would be willing to assume that $p_\theta(\cdot)$ would be known if the value of some uncertain parameter ϕ were known. That is, $p_{\theta|\phi}$ ($p_{\theta|\phi}(\cdot|\cdot)$ would be known.

For example, it might be assumed that θ is normally distributed with known variance σ_θ^2 and uncertain mean ϕ. In that case

$$p_{\theta|\phi}(\theta|\phi) = \frac{\exp\{-[\theta - p]^2/2\sigma_\theta^2\}}{\sigma_\theta\sqrt{2\pi}} \tag{6.2.18}$$

If $p_{\theta|\phi}(\cdot|\cdot)$ is assumed known then

$$U(x) = E_\phi\{K^{-1}\{\exp[nE_{\theta|\phi}\{\log_e[Kku(x|\theta)+1]|\phi\}]-1\}\}. \tag{6.2.19}$$

This is specified as soon as $p_\phi(\cdot)$ is assesed. The utility of any action a_k is

$$U(a_k) = E_{x|a}[E_\phi\{K^{-1}\{\exp[nE_{\theta|\phi}\{\log_e[Kku(x|\theta)+1]|\phi\}]-1\}\}|a_k]. \tag{6.2.20}$$

Although (6.2.20) is complicated algebraically, none of the operations needed to assess it are conceptually difficult. The numerical work needed can be carried out by computer.

Equation (6.2.20) can easily be updated using sample information. Suppose a sample S is selected using random sampling with replacement. If this consists of r values for θ, then

$$\underline{P}_{s|\phi}(S|\phi) = \prod_{i=1}^{r} p_{\theta|\phi}(\theta_i|\phi) \tag{6.2.21}$$

where $\theta_1, \theta_2, \ldots, \theta_r$ are the sample values and $P_{s|\phi}(S|\phi)$ is the probability of obtaining the sample results given in ϕ. Then, by Bayes' theorem,

$$p_{\phi|S}(\phi|S) = \frac{p_\phi(\phi)\prod_{i=1}^{r} p_{\theta|\phi}(\theta_i|\phi)}{\prod_{i=1}^{r}\int_\phi p_\phi(\tilde{\phi})p_{\theta|\phi}(\theta|\tilde{\phi})d\tilde{\phi}} \tag{6.2.22}$$

Thus the values of U(x) and $U(a_k)$ updated to account for S are

$$U(x|S) = E_{\phi|S}\{K^{-1}\{\exp[nE_{\theta|\phi}\{\log_e[Kku(x|\theta)+1]|\phi\}]-1\}|S\} \qquad (6.2.23)$$

and

$$U(a_k|S) = E_{x|a}[E_{\phi|S}\{K^{-1}\{\exp[nE_{\theta|\phi}\{\log_e[Kku(x|\theta)+1]|\phi\}]-1\}|S\}|a_k]. \qquad (6.2.24)$$

The results so far in this section have all involved situations where a scalar attribute x is sufficient to describe outcomes. In theory the discussion could be generalized to multiattribute situations. Additional parameters might be introduced to account for variations in the form of the multiattribute utility function from individual to individual. Thus, the form might be

$$u_i(\underline{x}) = u(\underline{x}|\theta_1,\theta_2,\ldots,\theta_n) \qquad (6.2.25)$$

where $\theta_1,\theta_2,\ldots,\theta_n$ are the parameters.

However, to use this formulation it would be necessary to assess the joint probability distribution over $\theta_1,\theta_2,\ldots,\theta_n$. Usually this would be very difficult. Hence it does not seem useful to extend the work in this section to the multiattribute case.

Part B

Decision Makers with Direct Preferences for Outcomes

6.3 General Comments

If the decision maker has preferences directly for outcomes $\underline{x}$ as well as for the utilities $u_1,u_2,\ldots,u_n$ then, as was noted at the beginning of this chapter, it will often be possible to write

$$U(\underline{x};\underline{u}) = K_1 U_x(\underline{x}) + K_2 U_u(\underline{u}) + K_3 U_x(\underline{x}) U_u(\underline{u}) \tag{6.3.1}$$

where $U_x(\underline{x})$ and $U_u(\underline{u})$ are conditional utility functions, and K_1, K_2 and K_3 are scaling constants. In this case, many of the results derived in Part A hold with only slight modifications. The nature of these modifications is indicated in the next two section.

6.3.1 Probabilistically Independent u_i's

If the u_i's are mutually probabilistically independent and if the $\underline{x}$ that results from any action a_k is known for certain then it follows from (6.3.1) that the utility of any action a_k is

$$U(a_k) = K_1 U_x(\underline{x}_k) + K_2 E_{u|x}[U_u(\underline{u}) | \underline{x}_k] + K_3 U_x(\underline{x}_k) E_{u|x}[U_u(\underline{u}) | \underline{x}_k] \tag{6.3.2}$$

If (6.0.2) holds then

$$U(a_k) = K_1 U_x(\underline{x}_k) + K_2 K^{-1}\{ \prod_{i=1}^{n} [K k_i E(u_i | \underline{x}_k) + 1] - 1\}$$
$$+ K_3 U_x(\underline{x}_k) K^{-1}\{ \prod_{i=1}^{n} [K k_i E(u_i | \underline{x}_k) + 1] - 1\}. \tag{6.3.3}$$

This equation is analogous to (6.1.3) which held in the case when U was not directly dependent on $\underline{x}$.

Suppose a sample S consisting of the unbiased utilities of the $j^{\underline{th}}$ individual or group is obtained. Then the updated value of U is

$$U(a_k) = K_1U_x(\underline{x}_k)+K_2K^{-1}\{[u_j(a_k)+1]\prod_{\substack{i=1\\i\neq j}}^{n}[Kk_iE(u_i|\underline{x}_k)+1]-1\}$$
$$+ K_3U_x(\underline{x}_k)K^{-1}\{[u_j(a_k)+1]\prod_{\substack{i=1\\i\neq j}}^{n}[Kk_iE(u_i|\underline{x}_k)+1]-1\}. \qquad (6.3.4)$$

This equation is analogous to (6.1.5) which held when U was not directly dependent on $\underline{x}$.

As these results indicate it is very easy to generalize the deviations for the case when $U = U(\underline{u})$ to $U = U(\underline{x};\underline{u})$ if $\underline{x}$ and $\underline{u}$ are mutually utility independent. The reader can easily do this for the situation where there is bias or where the $\underline{x}$ resulting from any a_k is uncertain.

6.3.2 Probabilistically Dependent u_i's

This case can also be solved easily. All of the results of section 6.2 hold in this situation if they are applied to the conditional utility function $U_u(\underline{u})$. This conditional utility function can then be combined with $U_x(\underline{x})$ using equation (6.3.1). Thus, for example, if the $\underline{x}$ resulting from any a_k is certain, then

$$U(a_k) = K_1U_x(\underline{x}_k) + K_2U_u(a_k) + K_3U_x(\underline{x}_k)U_u(a_k) \qquad (6.3.5)$$

where $U_u(a_k)$ is given by equation (6.2.9):

$$U_u(a_k) = K^{-1}E_{\theta|x}\{\exp[nE_{u|x,\theta}\log_e(Kku+1)|\underline{x}_k,\theta\}]-1|\underline{x}_k\} \qquad (6.3.6)$$

In the same way, if the $\underline{x}$ resulting from a_k is uncertain then

$$U(a_k) = K_1 E_{x|a}[U_x(\underline{x})|a_k] + K_2 E_{x|a}[U_u(\underline{x})|a_k] + K_3 E_{x|a}[U_x(\underline{x})\,U_u(\underline{x})|a_k] \tag{6.3.7}$$

where $U_u(\underline{x})$ is given by equation (6.2.19):

$$U_u(\underline{x}) = E_\phi\{K^{-1}\{\exp[nE_{\theta|\phi}\{\log_e[Kku(x|\theta)+1]|\phi\}]-1\}\}. \tag{6.3.8}$$

The reader can easily generalize the other results of section 6.2 to the case where U is dependent on $\underline{x}$ as well as $\underline{u}$.

* * * * * *

This concludes the discussion of uncertainty about the preferences of group members. It also concludes the development in the last three chapters of theory that is useful for decision analysis when the preferences of others are to be incorporated into the analysis. In the next chapter this theory is applied to three different situations.

Chapter VI Footnotes

1. See Raiffa and Schlaiffer[30] or Pratt, Raiffa and Schlaifer[27] for a detailed discussion of statistical decision theory.

2. See Pratt, Raiffa and Schlaifer[27], ch. 23B, for a general discussion of biased measurements.

3. See Schlaiffer[31], pp. 396-98, for a discussion of random sampling with replacement.

Chapter VII

APPLICATIONS

The results of the last three chapters are applied to three different situations in this chapter. In application A it is shown how the moderator of a discussion group could used decision analytic methods to determine and summarize the preferences of the group members. In particular, it is shown how these methods could be used to determine the preferences of community groups for different proposed government courses of action.

Application B shows how the preferences of the users of a computer time-share system might be determined for various system characteristics. Methods are given for combining the preferences of all the users into one measure of overall user preference for different system characteristics. This could be used by the time-share system manager as a guide to desirable improvements in the system.

Application C considers how the preferences for different types of housing of persons being displaced by highway construction could be assessed. In particular, their preferences for characteristics of possible sites for new replacement housing are studied. It is shown how these preferences might be determined and then analyzed to select a site that best meets the desires of the people being relocated.

The three applications presented here were undertaken because they illustrate well the strengths and weaknesses of the methodology developed in this thesis. In addition, the problems studied are currently of interest to analysts and researchers working in the fields.

Application A

Citizen Participation in Community Decision Making

7.1 Background

Sheridan[1] and Lemelshtrich[2] have studied methods for conducting meetings using electronic feedback mechanisms. Using their approach, each participant in the meeting has a small electronic box with switches or dials on it which may be used to signal his views to the person conducting the meeting. This discussion moderator can use the rapid feedback from the group to guide his conduct of the meeting and to quickly carry out votes on questions before the group.

Lemelshtrich suggests that this approach would be particularly valuable in helping to provide citizen participation in community decision making. He visualizes a procedure where a group would be selected from the community in a manner similar to the way juries are selected at present. This group would discuss various courses of action open to the community. During the discussion information would be presented by experts about the consequences of the different courses of action. Then the group would evaluate the proposals and report their evaluation to the community government and the general citizenry.

Lemelshtrich believes that the citizen group would provide valuable inputs to the government. He also suggests that this approach would help restore a sense of participation in community affairs to the general citizenry.

According to Lemelshtrich, the electronic feedback devices would be useful in this community participation process because they would allow rapid transfer of information during the discussion and also because they would provide anonymity to the group members so they would answer questions more honestly.

One important special case of citizen participation involves deciding which of a number of proposed projects are to be funded by the government. Lemelshtrich discusses in detail how the citizen group might consider the different projects and make recommendations about which ones to fund. His procedure for determining the preferences of the group seems reasonable but is not based on any basic principles for combining the views of individuals to obtain a group preference measure.

In the next section a method for doing this using the theory developed in the last three chapters is presented.

7.2 Decision Analytic Approach

Formally, the problem of interest here may be stated as follows: Suppose there are n individuals evaluating the m projects $p_1, p_2, \ldots, p_m$. Suppose the costs of these projects are $c_1, c_2, \ldots, c_m$ respectively, and the total amount of money available to be spent on $p_1, p_2, \ldots, p_m$ is T. Then it is desired to find the combination of projects most preferred by the group subject to the constraint that the total amount spent on the projects is less than or equal to T.

A simple example may clarify this. Suppose there are three projects under consideration. The first, p_1, is an experimental educational program costing \$500,000; the second p_2, is a program to improve community roads costing \$500,000; and the third, p_3, is an increase in the size of the police force costing \$200,000. Suppose the total money available for these projects is \$1 million. Then, using the notation of the last paragraph, c_1 = \$400,00, c_2 = \$500,000, c_3 = \$200,000 and T = \$1 million.

In decision analytic terms the problem may be stated as follows: Let $U = U(u_1, u_2, \ldots, u_n)$ be the utility function representing the preferences of the group as a function of the utilities of the group members. If $a_1, a_2, \ldots, a_r$ are the various feasible combinations of projects, then it is desired to find the a_k such that $U(a_k) = U[u_1(a_k), u_2(a_k), \ldots, u_n(a_k)]$ is maximized. (In the example discussed above the feasible combinations are: $a_1 = p_1$ only, $a_2 = p_2$ only, $a_3 = p_3$ only, $a_4 = p_1$ and p_2, $a_5 = p_1$ and p_3, $a_6 = p_2$ and p_3, and a_7 = no project. The problem is to find the a_k with the highest utility to the group.

If the u_i's are order-one mutually utility independent and order-two mutually preferentially independent with conditional utility functions linear in the u_i's, then, as shown in section 4.3.2, either

$$U = K^{-1}\left[\prod_{i=1}^{n}(K k_i u_i + 1) - 1\right] \qquad (7.2.1)$$

or

$$U = \sum_{i=1}^{n} k_i u_i \tag{7.2.2}$$

where $K, k_1, k_2, \ldots, k_n$ are constants.

The process of selecting group members from the community, as Lemelshtrich conceives it, would involve individuals selected to represent the community as a whole rather than special groups. Therefore, it is reasonable that U should be symmetric with respect to the u_i's--that is, everyone's preferences should be weighted equally. (See section 4.2 for further discussion of symmetry.) In this case either

$$U = K^{-1}\left[\prod_{i=1}^{n} (Kku_i+1)-1\right] \tag{7.2.3}$$

or

$$U = k \sum_{i=1} u_i \tag{7.2.4}$$

where K and k are constants.

As noted in section 4.5.1, k is arbitrary, however, K must be assessed. The value of K is subjective and may differ from decision maker to decision maker. A number of different individuals and groups may be interested in the preferences of the citizen group (e.g., various members of the community government and different citizen interest groups). Thus, it would be useful to display the group utilities $U(a_k)$ for the various alternatives for several different values of K. Each person interested in the group's preferences could then use the K which he feels is appropriate for his purposes.

7.2.1 Assessment of the u_i's

In order to use equation (7.2.3) or (7.2.4) it is necessary to assess the utilities $u_i(a_k)$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots,r$. The members of a citizen group will often be analytically unsophisticated and unfamiliar with probabilistic reasoning. Thus it is difficult for them to consider the probabilistic tradeoffs that are essential to any utility assessment.

The approach taken here is to highly structure the utility assessment problem so that only a few questions need to be asked to specify the utilities. This involves making several assumptions about the preferences of the individuals. It will be shown, however, that these are reasonable in many cases.

In what follows each individual's utility function for money will be assessed. Then he will assign a monetary value to each project. (This may differ from its cost.) These pieces of data will then be combined to obtain the individual's utility for each feasible combination of alternatives.

Each person is assumed to be constantly risk averse toward money so that

$$u_i(x) = A - (\operatorname{sgn} r_i)\, Be^{-r_i x} \qquad (7.2.5)$$

where A, B and r_i are constants, sgn r_i is the algebraic sign of r_i, and x is the quantity of money. (Constant risk aversion was studied in section 5.1.2. Notice that $u_i(x)$ is individual i's utility function for money spent by the community government rather than by himself. This is because the group is

considering proposed projects to be undertaken by the community government.)

The constant risk aversion assumption is made partly out of convenience. As noted in section 5.1.3, exponential utility functions provide close approximations to many utility functions actually observed in real-world assessments.

In addition, constant risk aversion would be reasonable in many cases that citizen groups would consider. The quantities of money that they would be considering (for example, the \$1 million in the example of the last section) may be relatively small compared to the total amount of money being spent by the community. Thus it would be reasonable to assume constant risk aversion over the range of x being considered since it is only a moderate perturbation in the total amount spent by the government.

In addition to assuming a exponential utility over money, it is reasonable to assume that the amounts 0 and T have the same utility for each individual in the group. These two amounts represent the two extreme possibilities--either <u>none</u> of the money is spent or <u>all</u> of the money is spent. If utility 0 is assigned to x = 0 and utility 1 is assigned to x = T then

$$u_i(x) = \frac{1 - e^{-r_i x}}{1 - e^{-r_i T}} \tag{7.2.6}$$

for i = 1,2,...,n.

One lottery must be considered by each individual to assess r_i. For example, each person might assess his certainty equivalent for a lottery with

a 50-50 chance of yielding either 0 or T. The responses, made through the electronic feedback devices, could then be displayed to the group and a discussion carried on to make sure that each person really understood the consequences of his answer. Following this, any changes that were desired would be made and then r_i, $i = 1,2,\ldots,n$ would be calculated for each person.

7.2.2 Assessment of Cash Values for a_k's

After assessing each person's utility for money, his monetary value $V_i(p_j)$, $i = 1,2,\ldots,n, j = 1,2,\ldots,m$, for each project would be determined. The group members would have difficulty determining their $V_i(p_j)$'s. However, presumably the group would have received data about the costs of other projects that have been carried out in the past. The members could compare the proposed projects to these and decide on the relative values of them. This would help them assess the $V_i(p_j)$'s.

As each $V_i(p_j)$ is assessed its values might be displayed to the group using the electronic feedback device. This display could be used as a basis for discussion that might help the members clarify their value judgments.

The assessment is continued making the assumption that the monetary value of any combination of projects is the sum of the values of each project in the combination. This assumes that the projects do not re-enforce or cancel each others' effects. For example, this would be reasonable for an experimental educational program and a program to improve community roads. These programs will nether help nor hinder each other so it is reasonable to assume the monetary value of the two programs in the sum of their individual monetary values.

On the other hand, if programs to improve the local narcotics squad and to institute a drug rehabilitation program were being considered, it might be reasonable to assume that the two programs together have a higher monetary value than the sum of their values alone.

Before proceeding, some useful notation is established. Suppose combination a_k consists of the programs $p_{k_1}, p_{k_2}, \ldots, p_{k_{n_k}}$. If it is assumed that the value of a combination of projects is the sum of their individual values, then

$$V_i^*(a_k) = \sum_{j=1}^{n_k} V_i(p_{k_j}) \tag{7.2.7}$$

for $i = 1, 2, \ldots, n$, where $V_i^*(a_k)$ is the monetary equivalent of a_k for the $i^{\underline{th}}$ individual.

However, there is a cost

$$C(a_k) = \sum_{j=1}^{n_k} C_{k_j} \tag{7.2.8}$$

associated with the combination a_k. Also there is some unspent money

$$M(a_k) = T - C(a_k) \tag{7.2.9}$$

that will be left over if a_k is selected. Thus the monetary equivalent $m_i(a_k)$, $i = 1, 2, \ldots, n$, for the $i^{\underline{th}}$ individual of the action "spend the money necessary and institute programs $p_{k_1}, p_{k_2}, \ldots, p_{k_{n_k}}$" is

$$m_i(a_k) = V_i^*(a_k) - C(a_k) + M(a_k)$$

$$= T + \sum_{j=1}^{n_k} V_i(p_{k_j}) - 2 \sum_{j=1}^{n_k} C(p_{k_j}). \qquad (7.2.10)$$

Hence the utility of a_k to the $i^{\underline{th}}$ individual is

$$u_i(a_k) = \frac{1 - e^{-r_i m_i(a_k)}}{1 - e^{-r_i T}} \qquad (7.2.11)$$

where $m_i(a_k)$ is given by (7.2.10). Thus, in view of equations (7.2.3) and (7.2.4), the utility of a_k to the group is either

$$U(a_k) = K^{-1}[\prod_{i=1}^{n} (Kku_i + 1) - 1] \qquad (7.2.12)$$

or

$$U(a_k) = k \sum_{i=1}^{n} u_i(a_k) \qquad (7.2.13)$$

where $u_i(a_k)$ is given by (7.2.11).

7.2.3 Concluding Remarks

In theory the procedure above obtains the preferences of the group for the various feasible combinations of projects. However, in practice there is a substantial amount of numerical computation to be carried out. If the citizen group consists of more than a few people it will be infeasible to do this by hand. One way it might be done would be to have the electronic feedback devices that each person holds attached directly to a computer. (This

might be a dedicated mini-computer or a time-share access to a larger computer.) The computer could then carry out any calculations needed and display the results.

Perhaps more important than this question of technical feasibility is the issue of whether decision analysis is an appropriate type of analysis for this problem. It was necessary to make a number of strong assumptions to carry out the analysis. These seem to limit the usefulness of the approach greatly. However, any form of analysis will make assumptions so that the analysis is tractable. The decision analytic approach has the advantage of making these explicit while some other types of analysis do not show their assumptions explicitly.

Lemelshtrich observes that one important purpose of the citizen groups is to provide a feeling of citizen participation in community government. He feels[3] this will not happen if the preferences of the group members are assessed in a sophisticated manner which they cannot understand. The decision analytic approach is probably such an approach.

This is a valid objection. Unless much time is spent explaining the approach (a formidable task if the group is mathematically unsophisticated), it will be a "black box" that obtains the group preferences in a manner that the group cannot understand. Thus the group members will not have a feeling of participation in the process.

However, if the main objective is to obtain good preference information rather than to provide a feeling of group participation, the approach outlined here seems to be useful.

Application B

Preferences of Time-share Computer Users

7.3 Background

Grochow has studied the preferences of time-share computer users for various levels of service of the time-share system.[4] He identified a number of goals associated with the level of service being provided by the system, and also identified measures of the extent to which each goal is met.

In particular, he concentrated on three goals for system performance: high availability of system, short response time to trivial requests, and short response time to compute-bound requests. He selected as measures of the degree to which these goals are met the following:

A = probability of successful login when the system is up,

R_t = real time to respond to "edit" requests,

and

R_c = real time to respond to "compile" requests

Grochow measured several time-share computer user's utility functions over A, R_t and R_c. Although he was able to make a number of utility independence assumptions about these attributes, he found that it still took about ten hours to assess a utility function for one individual. This is too time consuming for time-share system managers to assess the utilities of their users in order to determine what types of improvements would have the most value to the system users.

In the next section ways of approximately assessing the users' utility functions will be considered. The amount of work needed to do this is

substantially less than that needed to assess the utility function exactly.

In the section following that one methods of combining the user utilities to obtain an overall utility function for the group of users are discussed.

7.4 Approximate Assessment of User Utilities

Grochow argued that R_c should be utility independent of A for any given R_t for most time-share users. In addition, he showed that R_t should be utility independent of A x R_c. He showed that if these utility independence conditions hold than the utilities along the seven heavy lines shown in figure 7.1a are sufficient to determine $u(A,R_t,R_c)$ for all A,R_t and R_c. That is, seven conditional utility functions and the utilities of six points must be assessed. This is a formidable task, particularly when the person whose utility is being assessed is not familiar with decision analytic methods.

7.4.1 Parametric Dependence Conditions

Suppose that in addition to the utility independence conditions discussed in the last section it is assumed that A is parametrically dependent on R_c and R_t along the four heavy vertical lines in figure 7.1a, that R_c is parametrically dependent on A and R_t along the two top lines, and that R_t is parametrically dependent on A and R_c along the bottom line. Then it is easy to show by the methods used in chapter V that the utilities of any 13 of the 15 points shown in figure 7.1b are sufficient to completely specify $u(A,R_t,R_c)$ if the conditional parametric forms $u_A(A|\theta), u_t(R_t|\theta)$ and $u_c(R_c|\theta)$ are known.

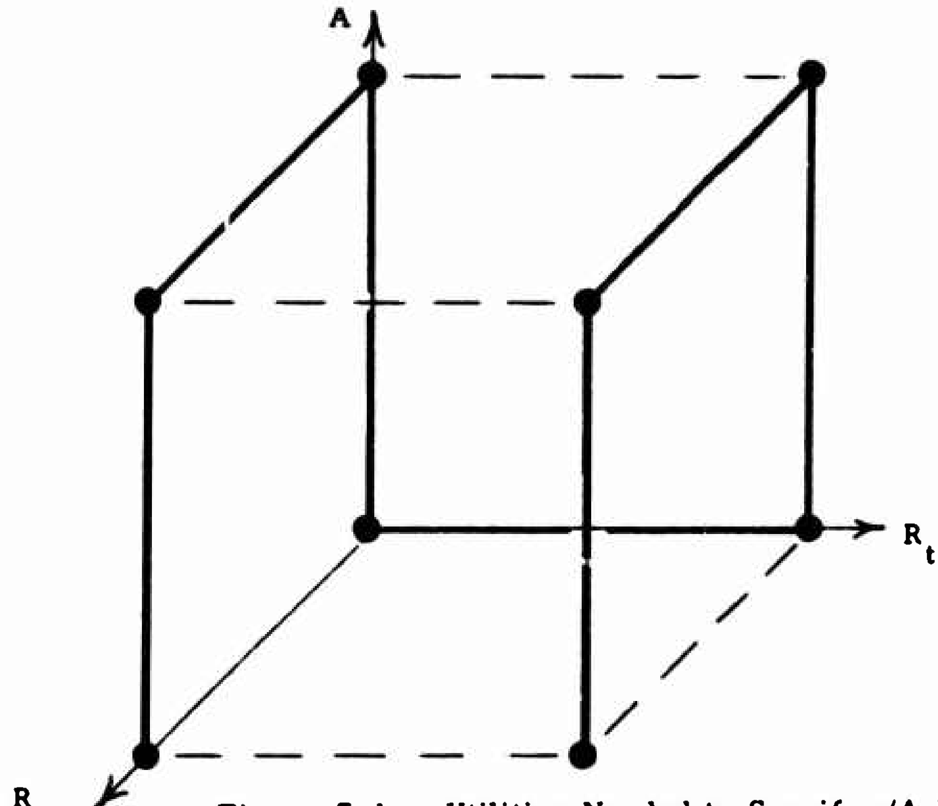


Figure 7.1a. Utilities Needed to Specify $u(A, R_t, R_c)$

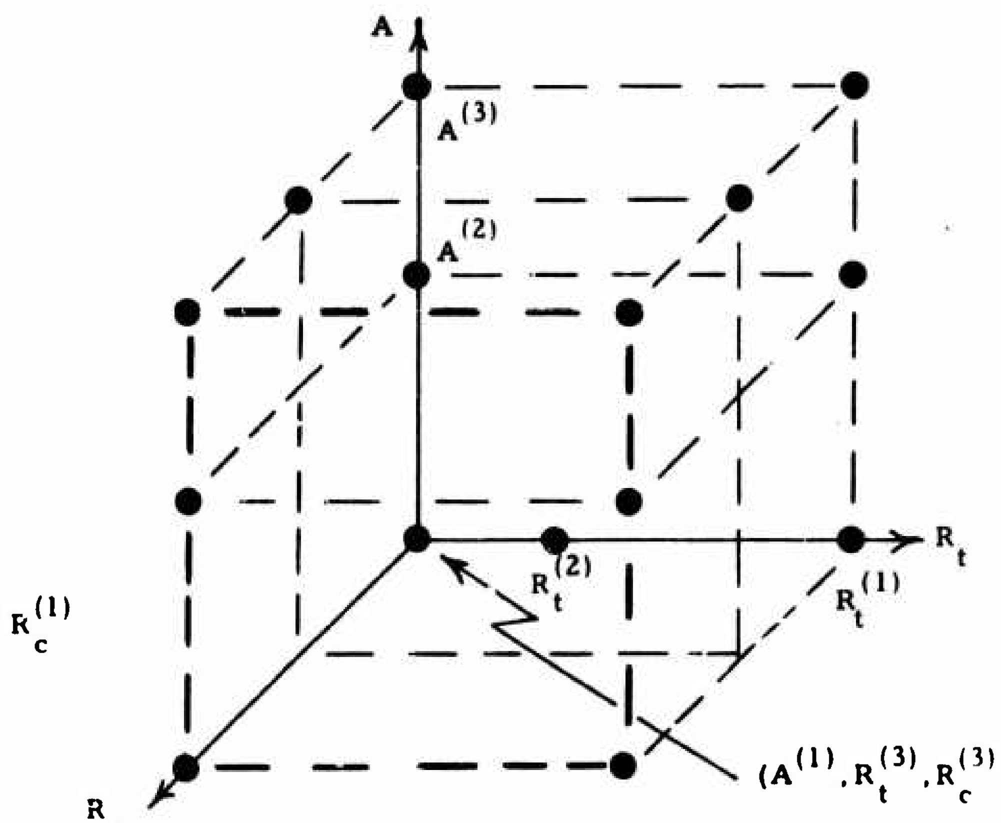


Figure 7.1b. Utilities Needed to Specify $u(A, R_t, R_c)$ with Parametric Dependence

Using the fact noted in section 5.1.3 that many empirically assessed utility functions can be adequately approximated by exponentials, it is reasonable to assume

$$u_A(A|\theta) = A_1 - B_1 e^{-\theta A}, \tag{7.4.1}$$

$$u_t(R_t|\theta) = A_2 + B_2 e^{\theta R_t}, \tag{7.4.2}$$

and

$$u_c(R_c|\theta) = A_3 + B_3 e^{\theta R_c} \tag{7.4.3}$$

where A_1, A_2, A_3, B_1, B_2 and B_3 are constants.

The difference between the form of (7.4.1) and that of the other two utility functions is due to the fact that greater values of A are more desirable while greater values of R_t and R_c are less desirable.

7.4.2 Assessment of Utilities

The approach taken to assessing the utilities of the points shown in figure 7.1b is to consider lotteries that are very similar to each other whenever possible. In this way, explicit consideration can easily be given to how the probabilistic tradeoffs change when only a few changes are made in a lottery.

First, find the p_{2ij} such that the decision maker is indifferent between receiving $(A^{(2)}, R_t^{(i)}, R_c^{(j)})$, i, j = 1, 3 for certain and receiving a lottery with probability p_{2ij} of obtaining $(A^{(3)}, R_t^{(i)}, R_c^{(j)})$ and probability $1 - p_{2ij}$ of

receiving $(A^{(1)}, R_t^{(i)}, R_c^{(j)})$. There are four such p_{2ij}'s. Notice, however, that the lotteries that must be considered are very similar. In all of them the only uncertainty is in A and this has the same form. Thus the decision maker can explicitly consider how his attitude toward this uncertainty depends on the fixed amounts of R_t and R_c.

When the p_{2ij}'s have been assessed, the utilities of the four center edge points on the sides of the cube in figure 7.1b will have been determined in terms of the eight corner utilities. In the same way, the utilities of the two top center edge utilities and the bottom center edge utility can be determined in terms of these corner utilities.

In the case of the top center edge points it is once again helpful that the lotteries that must be considered involve the same uncertainty in R_c with only a different amount of R_t. Thus the decision maker can conveniently consider how this change in R_t affects his preference for the uncertainty in R_c.

In order to complete the utility assessment the utilities of the eight corner points of the cube must be assessed. A procedure for doing this has been given by Raiffa.[5]

7.4.3 Practical Difficulties

Although there are no theoretical difficulties with the approach outlined in the last two sections, there is a lot of messy arithmetic that must be carried out. The utility assessments outlined in the last section must be used to determine the values of A_1, A_2, A_3, B_1, B_2 and B_3 in equations (7.4.1) -

(7.4.3) for the seven different conditional utility functions. Probably it would be necessary to use some type of computer analysis to do this.

7.5 Obtaining a Utility Function for the Users as a Group

In most cases the time-share system manager is not interested in the preferences of any one individual. He wishes to obtain a utility function representing the preferences of the group of users as a whole. This can then be used to determine the users' preferences toward various proposed changes in the system's operation.

If the preferences u_i, $i = 1,2,\ldots,n$ of the system users are order-one mutually utility independent and order-two mutually preferentially independent with conditional utility functions linear in the u_i's, then, as shown in section 4.3.2, either

$$U = K^{-1}\left[\prod_{i=1}^{n} (Kk_iu_i + 1) - 1\right] \tag{7.5.1}$$

or

$$U = \sum_{i=1}^{n} k_iu_i \tag{7.5.2}$$

where $k_1, k_2, \ldots, k_n$ and K are constants.

The following argument shows that often it would be reasonable for a time-share system manager to assume that the additive form (7.5.2) holds

Consider the lotteries

$$L_1 : \langle (u,o;u_{\overline{ij}});1/2;\ (o,u,u_{\overline{ij}}) \rangle$$

and

$$L_2 : \langle (u,u;u_{\overline{ij}});1/2,(o,o;u_{\overline{ij}}) \rangle \ .$$

It was shown in chapter IV that the additive form (7.5.2) holds only if the decision maker is indifferent between L_1 and L_2. It was argued there that often a decision maker would not be indifferent between L_1 and L_2 because in L_1 there is always a discrepancy between the utilities received by the two individuals while in L_2 they always receive equal utilities. If the decision maker is concerned about the "balance" of preferences in a group then he would not be indifferent between L_1 and L_2.

However, a time-share system manager might not be concerned with this balance. Usually time-share system users are physically separated from each other so they will not interact with each other and detect the lack of balance. Thus the time-share system will not lose any users due to this. Hence it is not of concern to the manager and (7.5.2) holds.

7.5.1 Consistent Scaling of the u_i's

Grochow noted that there is a level of service below which the system becomes essentially worthless to a user and also a level above which any increase in service has no added value because factors not related to the computer system limit use of the system. These levels of service differ for different individuals. As a practical approximation it is reasonable to assume that they have the same value to each user. Hence utility 0 could be assigned to the (A,R_t,R_c) below which the system is worthless to a particular user and utility 1

could be assigned to the (A, R_t, R_c) above which improvements are not useful.

7.5.2 Assessment of k_i's

Deciding on the values of the k_i's is difficult. Presumably the views of those people who use the system more should be counted more heavily than those who use it less. One way to do this would be to make k_i proportional to the amount of time the i^{th} individual uses the system, or, perhaps, the amount of money he spends on it.

A slightly more complicated procedure would be to make $k_i u_i$ proportional to the time used or money spent, where u_i is the utility of the present operating state to the i^{th} individual. This procedure recognizes that the utilization of the system by an individual may increase if the level of service increases. Thus, if a person is currently receiving a low level of service, his views should be weighted more (i.e., have a larger value of k_i) since an improvement in the perceived level of service could lead to an increase in his use of the system.

The two procedures above for evaluating the k_i's are <u>ad hoc</u> and open to criticism. However, since time-share users are a fairly homogeneous group it may be that their utility functions are relatively similar. In that case the details of the weighting procedure would not affect the final utility function for the group very much.

7.5.3 Concluding Remarks

The approach outlined in this application seems fairly practical. Time-share users usually are sympathetic toward quantitative approaches to problems. Probably fairly good utility assessments could be obtained from them.

A more important question is whether the detailed preference information that would be obtained using this approach is needed. Usually time-share managers use the system themselves and have a fairly good idea of what its strong points and weak points are without assessing the utilities of the users.

Application C

Assessing the Residential Preferences of Highway Relocatees

7.6 Background

The extensive highway construction in the U.S. during the last twenty years has led to the displacement of many people to make way for new highways. In many cases the people that must relocate are elderly or from minority groups and have limited financial means. These people often have difficulty finding housing comparable to that which they are forced to leave.

To alleviate this problem the Uniform Relocation Assistance and Land Acquisition Policies Act of 1970 was passed by Congress. This provides that highway departments can construct or rehabilitate replacement housing for highway relocatees if no housing comparable to what they are leaving is available. Highway departments must now decide whether they should construct new

housing, and if so, where it should be located and how it should be designed.

In order to make this decision the highway departments need information about the housing preferences of relocatees. Abt Associates, Inc. has designed a questionnaire to obtain preference information from highway relocatees.[6] In addition, it has developed a methodology to evaluate potential relocation plans in light of the preferences of the relocatees.

One portion of this methodology involv determining the preferences of the relocatees for different possible relocation sites and then deciding which ones are most preferred by the group c locatees. Although the method used by the Abt Associates analys's . ite complete, little or no theoretical justification is given for most of the steps in it.

In the next section an approach to solving this problem is given based on the theory developed in this thesis. Because of the complexity of the problem it will be necessary to make assumptions as the analysis proceeds in order to make it analytically tractable. However, the decision analytic approach makes these assumptions explicit. The Abt Associates method does not make clear what assumptions are made in the analysis.

Thus the decision analytic approach provides a framework which may be used to discuss the reasonableness of various assumptions that are made.

7.7 Decision Analytic Approach to Assessing Site Desirabilities

In the Abt Associates methodology information about the preferences of the relocatees is gathered by a "Housing Preference Questionnaire." (Copies of questions 15a, b and 33 from this questionnaire are included in Appendix 7.1.) Information about the relocatees

preferences for different site characteristics is gathered in questions 15a, b and 33b, c. Question 15 deals with preferences for convenience of various facilities, such as food stores and churches. Question 33 deals with preferences for different neighborhood characteristics, such as quietness and friendly neighbors.

Since the concern here is with selection of sites for construction of housing projects, only the responses to question 15 need to be considered. (The characteristics discussed in question 33 are relevant when discussing the detailed structure of the housing construction rather than the site location.)

The analysis in this section will use the data provided by the questionnaire in its current form. In section 7.8 a discussion will be given of ways the questionnaire might be modified to obtain better information about the preferences of the relocatees.

7.7.1 Assessing Individual Utilities

Suppose the distance to each facility is signified by the following:

x_1 = distance to food store,
x_2 = " " other shopping,
x_3 = " " hospital/clinic,
x_4 = " " church,
x_5 = " " public transport,
x_6 = " " elementary school,
x_7 = " " park or playground,
x_8 = " " day-care center,
x_9 = " " club/other social organization,

x_{10} = distance to local bar or restaurant, and

x_{11} = " " other entertainment.

Then the utility function of the $i^{\underline{th}}$ individual for closeness to facilities is given by

$$u_i = u_i(x_1, x_2, \ldots, x_{11}). \quad (7.7.1)$$

If the x_j's are assumed to be pair-wise preferentially independent and one of them is utility independent of the others then, as shown in section 4.0.1, either

$$u_i = K_i^{-1}\{\prod_{j=1}^{11}[K_i k_{ij} u_{ij}(x_j) + 1] - 1\} \quad (7.7.2)$$

or

$$u_i = \sum_{j=1}^{11} k_{ij} u_{ij}(x_j) \quad (7.7.3)$$

where k_{ij}, $j = 1, 2, \ldots, 11$ and K_i are constants. It will be assumed for analytic tractability that (7.7.3) holds.

The units of the x_j's must be specified. Since question 15 only asks for preferences concerning "nearness" to facilities (which is a subjective quantity) each x_i will be scaled from 0 to 1 where "$x_j = 0$" means the facility is next door and "$x_j = 1$" means the facility is far away. This is a subjective scale and the analyst may find it difficult to decide what the values of the x_j's are for a particular site. This problem will be discussed further below. Also, in section 7.8 a simple change in the questionnaire that would make the idea of "nearness" more clear will be considered.

If it is assumed for simplicity that the conditional utility function over each x_j is linear in x_j, then (7.7.3) reduces to

$$u_i = \sum_{j=1}^{11} k_{ij}(1 - x_j). \tag{7.7.4}$$

(Recall that greater values of x_j are less preferred than smaller values. This is why $u_{ij}(x_j) = 1 - x_j$ rather than $u_{ij}(x_j) = x_j$.)

The values of k_{ij}, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,11$ must now be determined. This can be done from the answers to questions 15a and b. There are six different levels of importance that a respondent may assign to being near to each type of facility:

1 = not so important,
2 = important,
3 = very important,
4 = third most important,
5 = second most important, and
6 = most important.

It will be assumed that the value of k_{ij} will be the $i^{\underline{th}}$ individual's assessment of importance as shown in the last paragraph. Thus if the $1^{\underline{st}}$ individual says it is "most important" for him to be near a food store and "important" to be near other shopping then $k_{11} = 6$ and $k_{12} = 2$.

This procedure has a number of deficiencies. In section 7.8 a simple change in the questionnaire that would improve it is considered.

7.7.2 Assessing the Group Preferences

In order to judge the desirability of a particular proposed site for new housing it is necessary to find the utility of the entire group of people that would live at the site. If the preferences u_i, $i = 1,2,\ldots,n$ of the various relocatees are order-one mutually utility independent and order-two mutually preferentially independent with conditional utility functions linear in the u_i's, then, as shown in section 4.3.2, either

$$U = K^{-1}\left[\prod_{i=1}^{n} (Kk_iu_i + 1) - 1\right] \tag{7.7.5}$$

or

$$U = \sum_{i=1}^{n} k_iu_i \tag{7.7.6}$$

where $k_1, k_2, \ldots, k_n$ and K are constants. The value of K should be assessed by the person responsible for deciding on the location of the housing site. (He could use the methods developed in section 4.5.1.) Since the decision maker would probably wish to treat the preferences of the individuals symmetrically, then (7.7.5) and (7.7.6) reduce to

$$U = K^{-1}\left[\prod_{i=1}^{n} (Kku_i + 1) - 1\right] \tag{7.7.7}$$

and

$$u = k \sum_{i=1}^{n} u_i \tag{7.7.8}$$

where K and k are constants.

This concludes the assessment of the group utility function for various values of $x_1, x_2, \ldots, x_{11}$. To use the assessed utility function to determine the relocatee group's relative utilities for different proposed relocation sites the values of $x_1, x_2, \ldots, x_{11}$ for each site would be determined. These would be used to determine the utilities for the individuals using equation (7.7.4), and these in turn would be used to calculate the group utility from equation (7.7.7) or (7.7.8).

As noted above, the assessment of the x_j's is subjective. Therefore, it does not make sense to use a very fine scale to specify the values of the x_j's for each proposed relocation site. For example, a three step scale might be used: $x_j = 0$ (facility very close), $x_j = 1/2$ (facility at a moderate distance), and $x_j = 1$ (facility far away).

7.8 Proposed Questionnaire Changes

Because of the lack of questions dealing with probabilistic tradeoffs on the Housing Preference Questionnaire it was necessary to make extensive assumptions about the form of the individual utility functions. Probabilistic tradeoff questions might be asked, however, these are often hard for interviewees to answer. If the interviewer is not skilled in asking such questions the answers obtained will often not be meaningful. The Housing Preference Questionnaire is designed to be administered by housing relocation specialists. These people will usually not be familiar with probabilistic tradeoff questions. Therefore, it does not seem useful to include such questions.

However, the utility assessment might be improved by the inclusion of two simple changes in the questionnaire. First, the subjectivity of the assessment of the x_j's could be decreased by including a question asking "how far would a facility have to be from your home before it became quite inconvenient."

Suppose the $i^{\underline{th}}$ individual responded "x_i^* miles." Then it would be reasonable to assume

$$u_ij(x_j) = \begin{cases} 1 - x_j/x_i^* , & x_j \leq x_i^* \\ \\ 0, & \text{otherwise.} \end{cases} \tag{7.8.1}$$

The different values of x_i^* for different individuals would account for items like possession of an automobile or different abilities to walk due to different states of health.

If (7.8.1) is used then the analyst no longer needs to assess a subjective measure of how far each facility is from the proposed relocation sites that are being evaluated. He can measure their actual distances and substitute this into (7.8.1).

Another feature of the assessment that could be improved by a simple change in the questionnaire is the assessment of the scaling constants in equation (7.7.4). The procedure given in the last section guarantees that for every individual there will be a $k_{ij} = 6$ and also k_{ij}'s equal to 5 and 4. This does not seem reasonable since different individuals will often have different

preferences for facilities being convenient to their home. For example, a retired person without a car might have a strong preference for convenient facilities. On the other hand, a working person who drives into the business district every day might not be very concerned with convenience of facilities since he would have access to many of them in the business district.

To gain some measure of this difference questions 15a and b might be combined into one question which asks the individual to rate the importance of having each facility convenient on a 1 to 6 scale where 1 means "unimportant" and 6 means "extremely important."

Using this procedure those to whom facility location was important would rate importance of convenient facilities high for all facilities. Those to whom facility location was unimportant would rate it low for all facilities.

The rating numbers would still be used as the k_{ij}'s just as in section 7.7. Now, however, these might be a more accurate indication of individual preferences than they were before.

7.9 Concluding Remarks

A large number of assumptions had to be made to apply the methods developed in this thesis to the relocation analysis. However, at least as many assumptions must be made to use other forms of analysis. The decision analytic approach has the advantage that it makes the assumptions explicit so that the weaknesses of the study are pointed out. As shown in the last section, this can sometimes help to uncover simple changes that will make the analysis more accurate.

The comments above seem to apply to all three of the applications in this chapter. The methods developed in this thesis provide a framework for incorporating the preferences of others into an analysis. A number of assumptions must be made to apply this framework to any particular situation. However, these are no more extensive than with other forms of analysis and this approach has the advantage of making them explicit.

Housing Preference Questionnaire

15 People who live in the city are often concerned about the convenience of certain facilities in their neighborhood. Now I'd like to ask you a few questions about the places that are important to people living in this (house / apartment).	a Is it very important (3), important (2), or not so important (1) for the people in this (house / apartment) to be very close to the following (READ LIST AT LEFT).	b If you could choose only three to be near, which would be the most important? (6), the second most important? (5), the third? (4).
food store	13	14
other shopping	20	21
hospital/clinic	27	28
church SPECIFY	34	35
public transport	41	42
elementary school	48	49
park or playground	55	56
day-care center	62	63
club/other social organization SPECIFY	69	70
local bar or restaurant	76	77
other entertainment SPECIFY	14	15
anywhere else SPECIFY	21	22

Housing Preference Questionnaire (continued)

33a Now I'd like to ask you a few questions about your neighborhood. First, how long have you lived in this neighborhood?
CODE NUMBER OF YEARS. 71–72

b Now I'd like to read you a list of things which may be important considerations for someone choosing a place to live. I'd like you to tell me which are very important (3), which are important (2), and which are not so important (1) for this household. I'll read through the list first, and then you can decide. READ ENTIRE LIST. THEN RETURN TO ITEM 1 AND REPEAT, RECORDING EACH RESPONSE.		c If you could choose only three, which would be the most important? (6), the second most important? (5), the third? (4). IF NECESSARY, REPEAT THE LIST OF "VERY IMPORTANT" AND "IMPORTANT" ITEMS.
a quiet neighborhood	73	74
friendly neighbors	75	76
a good neighborhood reputation	77	78
neighbors with ethnic backgrounds similar to yours	79	80
neighbors with education similar to yours	11	12
good police protection	13	14
good fire protection	15	16
frequent garbage collection	17	18
lots of parks and green space	19	20
well-maintained streets	21	22
well-maintained houses and yards	23	24
easy access to other places	25	26
good school district	27	28
street lights	29	30

Chapter VII Footnotes

1. See Sheridan[32].

2. See Lemelshtrich[21].

3. Private communication.

4. See Grochow[9].

5. See Raiffa[29].

6. This work was performed for the Federal Highway Administration under contract number FH-11-7527. Abt Associates, Inc. is a social science research and consulting firm located in Cambridge, Mass. The results of the study are reported in Abt Associates, Inc.[1,2].

Chapter VIII

SUGGESTIONS FOR FURTHER RESEARCH

The suggestions for further research related to this thesis fall into two areas: applications oriented and theoretical. In the applications area, more experience with applying the methods developed here is needed. This will bring into sharper focus the strengths and weaknesses of the decision analytic approach to incorporating the preferences of others into an analysis.

In particular, more experimentation is needed to see how widely applicable the parametric dependence conditions studied in chapter V are. Also, more work is needed applying the approximate methods for dealing with uncertainty discussed in chapter VI. This work should point out areas where the methods could be improved.

Additionally, more experience is needed in assessing the scaling constants for $U(\underline{x};\underline{u})$. This was discussed in section 4.5 but additional research should lead to improved procedures for finding these constants.

On the theoretical side, the most promising area of research involves the interdependent utility functions discussed in section 5.4. In particular, it seems that useful results could be obtained by studying arbitration schemes, such as the Nash solution, using interdependent utility functions.

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